Abstract Algebra Prelim Jan. 2014

1. If $g \in G$ has order m, $h \in G$ has order n, m and n are relatively prime and $gh = hg$, prove gh has order mn.

2. Show that there are only two groups of order 21 up to isomorphism, using semidirect products.

3. Let G be a finite abelian group, H a subgroup of G, and $\chi\colon H \to \mathbf{C}^\times$ a character on H. Show χ can be extended to a character on G.

4. Let A and B be commutative rings. Show every ideal in the ring $A \times B$ has the form $I \times J$ where I is an ideal in A and J is an ideal in B. Hint: $(1,0)(x,y) = (x,0)$.

5. (a) Show $F[x]$ is a Euclidean domain, where F is a field.

   (b) Show every Euclidean domain is a PID.

6. Give examples as requested, with brief justification.

   (a) An infinite abelian group in which every element has finite order.

   (b) A permutation $\pi \in S_5$ such that $\pi(12)(345)\pi^{-1} = (35)(124)$.

   (c) A maximal ideal M in $\mathbf{Z}[x]$ such that $\mathbf{Z}[x]/M$ has order 25.

   (d) A Euclidean domain other than $\mathbf{Z}$ or $F[x]$ or F (F a field).