Abstract Algebra Prelim Jan. 2012

1. (a) Define a p-Sylow subgroup of a finite group.

   (b) For each prime p, prove that any two p-Sylow subgroups of a finite group are conjugate. (That is, prove the second part of the Sylow theorems.)

2. Let the additive group $\mathbf{Z}$ act on the additive group $\mathbf{Z}[\frac{1}{3}] = \{a/3^k : a \in \mathbf{Z}, k \geq 0\}$ by $\varphi_n(r) = 3^n r$ for $n \in \mathbf{Z}$ and $r \in \mathbf{Z}[\frac{1}{3}]$. Set $G = \mathbf{Z}[\frac{1}{3}] \rtimes_\varphi \mathbf{Z}$, a semi-direct product.

   (a) Compute the product $(r, m)(s, n)$ and the inverse $(r, m)^{-1}$ in the group G.

   (b) Show G is generated by $(1, 0)$ and $(0, 1)$.

3. Let R be a ring with identity, possibly noncommutative. Let I and J be *two-sided* ideals in R. Define IJ to be the set of finite sums $a_1b_1 + \cdots + a_nb_n = \sum_{k=1}^n a_kb_k$ where $n \geq 1$, $a_k \in I$, and $b_k \in J$.

   (a) Prove that IJ is a two-sided ideal in R and that $IJ \subset I \cap J$.

   (b) If R is *commutative* and $I + J = R$ then prove $IJ = I \cap J$, indicating where you use the commutativity in your proof.

   (c) Let $R = \left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Z} \\ 0 & \mathbf{Z} \end{smallmatrix}\right) = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) : a, b, c \in \mathbf{Z}\}$, which is a noncommutative ring under addition and multiplication of matrices. Set

   $$I = \left(\begin{smallmatrix} 0 & \mathbf{Z} \\ 0 & \mathbf{Z} \end{smallmatrix}\right) = \{\left(\begin{smallmatrix} 0 & y \\ 0 & z \end{smallmatrix}\right) : y, z \in \mathbf{Z}\} \quad \text{and} \quad J = \left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Z} \\ 0 & 0 \end{smallmatrix}\right) = \{\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) : x, y \in \mathbf{Z}\}.$$

   Show I and J are two-sided ideals in R, $I + J = R$, and $IJ \neq I \cap J$. (This shows that part b becomes false in general if we drop its commutativity hypothesis.)

4. (a) Show the only units in $\mathbf{Z}[\sqrt{-5}]$ are ± 1.

   (b) Define what it means for an integral domain R to be a *unique factorization domain* (UFD) and use the equation $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ to show $\mathbf{Z}[\sqrt{-5}]$ is not a unique factorization domain.

5. Let R be a commutative ring. Show a nonzero ideal I in R is a free R-module if and only I is a principal ideal with a generator that is not a zero divisor in R. (Hint: For the direction $(\Rightarrow)$, show a basis of I can't have more than one term in it.)

6. Give examples as requested, with brief justification.

   (a) A group action which has no fixed points.

   (b) The class equation for a non-abelian group that is not isomorphic to S_3. (Be sure to specify what the group is.)

   (c) A cyclic $\mathbf{R}[X]$-module that is three-dimensional as a vector space over $\mathbf{R}$.

   (d) A unique factorization domain (UFD) which is not a principal ideal domain (PID).