1. Let $R$ be a commutative ring with identity, and let $M$ be an $R$-module. Recall the annihilator of $M$ is $\text{Ann}(M) = \{ r \in R \mid rm = 0 \text{ for all } m \in M \}$. For any ideal $I$ in $R$, show $M$ is an $R/I$-module by the rule $(r + I) \cdot m = rm$ if and only if $I \subseteq \text{Ann}(M)$.

2. Let $R$ be a commutative ring with identity, and let $I$ and $J$ be ideals in $R$. Recall that $I + J = \{ r + r' \mid r \in I, r' \in J \}$, and $IJ$ is the ideal generated by all products $rr'$ with $r \in I$ and $r' \in J$.

   (a) Prove that if $I + J = R$ then $IJ = I \cap J$.

   (b) Assuming that $I + J = R$, show that for any $a$ and $b$ in $R$ there exists some $x \in R$ such that $x \equiv a \mod I$ and $x \equiv b \mod J$. (Recall that $x \equiv a \mod I$ if and only if $x - a \in I$.)

3. Let $\varphi : \mathbb{Z} \to \text{Aut}(\mathbb{Z})$ by $n \mapsto \varphi_n$, where $\varphi_n(a) = (-1)^n a$. Define the semi-direct product group $G = \mathbb{Z} \rtimes \varphi \mathbb{Z}$.

   (a) Write down the group law and the formula for inverses in $G$.

   (b) Find the center of $G$.

4. In a commutative ring $R$, an ideal $Q$ is called primary if whenever any $a$ and $b$ in $R$ satisfy $ab \in Q$ and $a \notin Q$, we have $b^n \in Q$ for some integer $n \geq 1$. (Equivalently, if $ab \equiv 0 \mod Q$ and $a \not\equiv 0 \mod Q$, we have $b^n \equiv 0 \mod Q$ for some integer $n \geq 1$. That is, in the ring $R/Q$ any zero divisor is nilpotent.) Show that the nonzero primary ideals in a PID are the ideals of the form $(p^n)$ where $p$ is a prime element and $n$ is a positive integer. You may use that a PID is a UFD.

5. In $\mathbb{R}^3$ a line-plane pair is a pair of subspaces $(V_1, V_2)$ where $V_1 \subset V_2$, $\dim V_1 = 1$, and $\dim V_2 = 2$. The standard line-plane pair in $\mathbb{R}^3$ is $(R e_1, R e_1 + R e_2)$ where $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. Let $S$ be the set of all line-plane pairs in $\mathbb{R}^3$.

   (a) The group $\text{GL}(3, \mathbb{R})$ of invertible $3 \times 3$ real matrices acts on $S$ by

   $$A \cdot (V_1, V_2) = (A(V_1), A(V_2)),$$

   where $A \in \text{GL}(3, \mathbb{R})$ and $(V_1, V_2) \in S$. Prove that the stabilizer subgroup of the standard line-plane pair is the group of invertible upper-triangular matrices in $\text{GL}(3, \mathbb{R})$ (with arbitrary non-zero entries on the diagonal).

   (b) Prove that the $\text{GL}(3, \mathbb{R})$-action on $S$ is transitive.

6. Give examples as requested, with brief justification.

   (a) A maximal ideal in $\mathbb{C}[x, y]$ which contains the ideal $(xy, x^2 - 1)$.

   (b) A ring $R$ and ideals $I$ and $J$ in $R$ such that $IJ \neq I \cap J$.

   (c) A generator of the group of characters of $(\mathbb{Z}/7\mathbb{Z})^\times$.

   (d) A finite nonzero $\mathbb{Z}[i]$-module.