1. Prove the rings $\mathbb{Z}/mn\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ are isomorphic when $m$ and $n$ are relatively prime (positive) integers. Discuss whether these rings are ever isomorphic when $m$ and $n$ are not relatively prime.

2. Let $S = \{(z, w) \in \mathbb{C} \times \mathbb{C} : |z|^2 + |w|^2 = 1\}$. For a positive integer $m$, let $\mathbb{Z}/m\mathbb{Z}$ act on the set $S$ by

$$(a \mod m) \cdot (z, w) = \left(e^{2\pi ia/m}z, e^{8\pi ia/m}w\right).$$

(a) Show this is a group action of $\mathbb{Z}/m\mathbb{Z}$ on $S$.

(b) If $m$ is odd, show every orbit in this group action has $m$ elements.

(c) If $m$ is even, show the orbit of some point in $S$ has less than $m$ elements.

3. Use Zorn’s lemma to show every nontrivial finitely generated group contains a maximal subgroup. (A maximal subgroup is a proper subgroup contained in no other proper subgroup.) Do not assume the group is abelian.

4. (a) Let $a$ be any complex number. Prove that the map $\phi: \mathbb{R}[x] \to \mathbb{C}$ defined by $\phi(f(x)) = f(a)$ is a homomorphism of rings.

(b) Prove that $\mathbb{R}[x]/(x^2 + 1)$ is a field which is isomorphic to $\mathbb{C}$.

5. (a) Let $R$ be a commutative ring with identity and $I$ be an ideal in $R$. Show that $R/I$ is a field if and only if $I$ is a maximal ideal.

(b) Let $R$ be a PID and $P$ be a nonzero prime ideal in $R$. Show that $P$ is a maximal ideal.

6. Give examples as requested, with brief justification.

(a) A nonabelian group which is not isomorphic to a semidirect product of nontrivial groups.

(b) A 2-Sylow subgroup of $S_4$.

(c) A PID other than $\mathbb{Z}$.

(d) A unit other than $\pm 1$ in $\mathbb{Z}[\sqrt{7}]$. 