Abstract Algebra Prelim Aug. 2010

1. Set $\mathbf{Z}[\sqrt{11}] = \{a + b\sqrt{11} : a, b \in \mathbf{Z}\}$.

   (a) Show $a + b\sqrt{11}$ is a unit in $\mathbf{Z}[\sqrt{11}]$ if and only if $a^2 - 11b^2 = \pm 1$.

   (b) Show there is no integral solution to $a^2 - 11b^2 = -1$.

2. Let G be an abelian group and H be a subgroup with finite index. For any integer $n \geq 1$, the inclusion $H \to G$ and the reduction $G \to G/nG$ compose to give a homomorphism $H \to G/nG$. This homomorphism kills nH, so it induces a homomorphism $f_n \colon H/nH \to G/nG$ given by $f_n(h \bmod nH) = h \bmod nG$.

   (a) Whenever $(n, [G : H]) = 1$, show f_n is an isomorphism.

   (b) Whenever $(n, [G : H]) > 1$, show f_n is not surjective.

3. Let S_n be the symmetric group on n letters ($n \geq 2$). Then every element of S_n can be written as a product of cycles.

   (a) Write an arbitrary cycle $(i_1\, i_2 \ldots i_m)$, $m \geq 2$, as a product of transpositions.

   (b) Show that S_n is generated by the $n-1$ transpositions $(12), (13), \ldots, (1n)$.

   (c) Show that S_n is generated by the $n-1$ transpositions $(12), (23), \ldots, (n-1\ n)$.

4. (a) In $\mathbf{Q}[x, y]$, prove that (x) is a prime ideal and not a maximal ideal.

   (b) In $\mathbf{Q}[x, y]$, prove that (x, y) is a maximal ideal.

   (c) In $\mathbf{Q}[x]$, prove that $(x^2 - 2)$ is a maximal ideal and that neither (x^2) nor $(x^2 - 4)$ is a prime ideal.

5. Let A be a nonzero ring such that $a^2 = a$ for all $a \in A$. (Examples include $\mathbf{Z}/2\mathbf{Z} \times \cdots \times \mathbf{Z}/2\mathbf{Z}$, but these are not the only ones.)

   (a) Show A has characteristic 2.

   (b) If A is finite, show its size is a power of 2.

   (c) Show any prime ideal in A is maximal.

6. Give examples as requested, with brief justification.

   (a) Four nonisomorphic groups of order 8.

   (b) A nontrivial character of the group $(\mathbf{Z}/12\mathbf{Z})^\times$.

   (c) A torsion-free $\mathbf{Z}$-module which is not a free $\mathbf{Z}$-module. (Torsion-free means no element v satisifes $nv = 0$ for some nonzero integer n except for $v = 0$.)

   (d) Three nonisomorphic $\mathbf{C}[x]$-modules which are each 2-dimensional as $\mathbf{C}$-vector spaces.