1a) Show that a group of order 45 is abelian.

1b) Is every group of order $p^2q$, with $p$ and $q$ distinct primes, abelian?

2) Suppose $G$ is a group. Recall that $G$ satisfies the ascending chain condition on subgroups, if for any subgroups $H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots$ there is a positive integer $i$ such that $H_i = H_{i+1} = H_{i+2} = \cdots$. Also $G$ satisfies the descending chain condition on subgroups if for any subgroups $H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$ there is an integer $i$ such that $H_i = H_{i+1} = H_{i+2} = \cdots$.

(a) Show that every finitely generated abelian group satisfies the ascending chain condition.

(b) Which finitely generated abelian groups satisfy the descending chain condition?

3) Let $R = \mathbb{Z}[x, y]$.

(a) If $I$ is a principal ideal of $R$, show that there are only finitely many principal ideals of $R$ which contain $I$.

(b) Show that $(x, y)$ is a prime ideal of $R$ which is not maximal.
4) Let $V$ be a vector space over the field $F$ and $B_1$ and $B_2$ two bases for $V$. Show that if $B_1$ has infinite cardinality then $B_2$ also has infinite cardinality. (You may not just quote the uniqueness of dimension for a vector space.)

5) Let $n$ be a positive integer, $GL(n, C)$ the group of invertible $n \times n$ complex matrices, and $G$ a finite group. Suppose $\Psi : G \rightarrow GL(n, C)$ is a group homomorphism. If $g \in G$, show that $\Psi(g)$ is a diagonalizable matrix.

6) Suppose $R$ is a commutative ring with 1, $M$ an $R$-module and 

$$\Psi : M \rightarrow R$$

an onto $R$-module homomorphism. Show that 

$$M = \text{Ker}(\Psi) \oplus B$$

for some submodule $B \subseteq M$ with $B \cong R$. 