315 Algebra Prelim Exam August 2004

1) Suppose G is a finite group, p a prime, A a p-Sylow subgroup of G and B a subgroup of G.

(i) If A is a normal subgroup of G, show that $A \cap B$ is a p-Sylow subgroup of B.

(ii) If B is a normal subgroup of G, show that $A \cap B$ is a p-Sylow subgroup of B.

2) Suppose G is a finite group and A the subgroup generated by $\{x^3 \mid x \in G\}$.

(a) Show that A is a normal subgroup of G.

(b) Show that any element in G/A is of order dividing 3.

(c) Suppose B is a normal subgroup of G with the property that any element of G/B is of order dividing 3. Show that $B \supseteq A$.

3) Let V be a finite dimensional vector space and W a subspace of V. Then there is a linear transformation $\phi : V \longrightarrow V$ such that $\phi^2 = \phi$ and $\phi(V) = W$.

over

4) Let $R = Z[x]$, be the ring of polynomials with integer coefficients. Tell if each of the following are true or false and give a reason.

(a) R is a principal ideal domain.

(b) Every prime ideal in R is maximal.

(c) R is a unique factorization domain.

(d) If $f, g \in R$ have greatest common divisor d, then there exist $u, v \in R$ with $d = uf + vg$.

5) If A, B are finitely generated abelian groups and $Z \oplus A \simeq Z \oplus B$, then $A \simeq B$. (Here, Z is the group of integers under addition.)

6) Let R be an integral domain.

(a) Suppose M is an R-module. Define "M is a free R-module".

(b) Let I be an ideal of R. Show that I, considered as an R-module, is a free R-module if and only if I is a principal ideal.