Algebra Preliminary Examination August 2003

1. Let R be a commutative ring with identity and A an ideal of R. Define

$$r(A) = \{x \in R \mid x^n \in A \text{ for some positive integer } n\}.$$

(i) If A is an ideal of R, show that $r(A)$ is an ideal of R.

(ii) If A is a prime ideal of R, show that $r(A) = A$.

(iii) If A, B are ideals of R, show that $r(A \cap B) = r(A) \cap r(B)$.

(iv) If $R = Z$ and A is the ideal generated by $a = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, with the p_i distinct primes and e_i positive integers, what is $r(A)$?

2. Suppose R is a ring with identity and M a left Artinian R-module. If $\phi : M \longrightarrow M$ is a one-to-one R-module homomorphism, show that ϕ is onto.

(Recall that M is Artinian if any decreasing chain of submodules $N_1 \supseteq N_2 \supseteq \cdots$ has the property that $N_i = N_{i+1}$ for some positive integer i.)

3. Let p and q be distinct primes. Show that a group of order p^2q has a normal Sylow subgroup.

4. Let n be a positive integer greater than 1 and R the ring of $n \times n$ matrices over a field. For $1 \leq j \leq n$, let L_j denote the matrices with arbitrary elements in the j-th column and zeros elsewhere.

(i) Show that L_1 is a minimal left ideal of R; that is L_1 is a nonzero left ideal of R that properly contains no other nonzero left ideal.

(ii) Show that L_1 is isomorphic as a left R-module to L_j for each $2 \leq j \leq n$.

(iii) Let L be any minimal left ideal of R. Show that L is R-module isomorphic to L_1.
Hint: Consider the ideals $LL_1, LL_2, \ldots, LL_n$.

5. Let G_1 and G_2 be two finite abelian groups with the property that for each integer $n > 1$, G_1 and G_2 have the same number of elements of order n. Show that G_1 and G_2 are isomorphic.

6. Let F be a field and $R = F[x_1, x_2, \ldots]$ the ring of polynomials over F in an infinite number of variables. Show that R is a unique factorization domain. Carefully state any theorems that you use.