1) Let $V$ be a finite dimensional vector space and $W_1, W_2$ subspaces of $V$. Show that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Here $W_1 + W_2 = \{w_1 + w_2 \mid w_i \in W_i\}$.

2) (i) Show that there does not exist a simple group of order 30.

(ii) Let $G$ be a finite group, $p$ a prime, and $P$ a normal $p$-Sylow subgroup of $G$. If $\phi : G \to G$ is a homomorphism, show that $\phi(P) \subseteq P$.

3) Let $R$ be a commutative ring with identity. Call $R$ a $*$-ring if the intersection of all non-zero ideals of $R$ is non-zero. (The ring $R$ itself is an ideal.)

(i) Let $\mathbb{Z}_n$ denote the ring of integers modulo $n$. Determine, with proof, those values of $n$ when $\mathbb{Z}_n$ is a $*$-ring.

(ii) If $R$ is an integral domain $*$-ring, show that $R$ is a field.
4) Let $A$ be a finitely generated infinite abelian group, and $n$ a positive integer. Show that there is a subgroup $B$ of $A$ with $|A/B| = n$.

5) Let $R$ and $S$ be integral domains with $R \subseteq S$. Suppose that $R$ is a PID. If $d$ is the greatest common divisor of $a$ and $b$ in $R$, show that $d$ is the greatest common divisor of $a$ and $b$ in $S$.

6) Let $R$ be a commutative ring with identity and consider the following commutative diagram of $R$-modules and $R$-module homomorphisms:

$$
\begin{array}{cccccc}
0 & \to & A & \overset{\alpha}{\to} & B & \overset{\beta}{\to} & C & \to & 0 \\
\theta_1 \downarrow & & \theta_2 \downarrow & & \theta_3 \downarrow \\
0 & \to & A' & \overset{\alpha'}{\to} & B' & \overset{\beta'}{\to} & C' & \to & 0
\end{array}
$$

Here the rows are exact, meaning that $\alpha, \alpha'$ are monomorphisms, $\beta, \beta'$ are epimorphisms, $\text{Im}(\alpha) = \text{Ker}(\beta)$, and $\text{Im}(\alpha') = \text{Ker}(\beta')$.

Given that $\theta_1$ and $\theta_3$ are isomorphisms, show that $\theta_2$ is an isomorphism.