Definition 1 *A left R-module A is **faithful** if, for all finitely generated right R-modules $M \neq 0$, $M \otimes_R A \neq 0$. If we can omit the finitely generated restriction, the module ${}_RA$ is called **fully faithful**.*

Now let A be a right W-module with endomorphism ring $R = End(A_W)$, acting on the left. We are interested in determining when the module ${}_RA$ is either faithful or fully faithful. The motivation here lies in the equivalence $1 \sim 2$ in the following theorem, given after one prerequisite definition.

Definition 2 *For A, G right W-modules, let $S_A(G)$, called the A-socle of G, be the W-submodule of G generated by the images of all maps from A into G; that is, $S_A(G) = \{\sum f(A) : f \in Hom_W(A, G)\}$.*

Theorem 3 *TFAE for A, R as above:*

1) The module ${}_RA$ is faithful.
2) Every exact sequence $0 \to K \to G \to A^n \to 0$, $0 < n < \omega$, with $K + S_A(G) = G$ splits. (Note that the equality $K + S_A(G) = G$ will always be necessary for the splitting of any such sequence.)
3) Let $F = (R_R)^m$ for some natural number n, and let I be a right ideal of $End(F_R)$ such that $I(F \otimes_R A) = F \otimes_R A$. Then $IF = F$.

Condition 2) is called Baer's lemma, first stated circa 1940 by Reinhold Baer for $G \leq Q^n$ and $A \leq Q$. The equivalence $1 \sim 2$ was first explicitly noted by Albrecht in the 1980's. In the mid 1970's, Arnold and Lady introduced these notions with 1) replaced by the equivalent condition 1') $IA \neq A$ for all right ideals $I \neq R$. Condition 3 was introduced in 1995 by Reid.

If the module A_W is **self-small** ($Hom(A, A^\alpha) = R^\alpha$ for any cardinal α), then we have an infinite version of Theorem 3, due to Albrecht and Reid.

Theorem 4 *TFAE for $A_W, R = End_W(A)$:*

1) The module ${}_RA$ is fully faithful.
2) Every exact sequence $0 \to K \to G \to A^\alpha \to 0$, with $K + S_A(G) = G$ splits.
3) Let $F = (R_R)^\alpha$ for some cardinal α, and let I be a right ideal of $End(F)$ such that $I(F \otimes_R A) = F \otimes_R A$. Then $IF = F$.

It is perhaps natural, in view of the finite character of the tensor product, to assume that faithful implies fully faithful. But, in 1994, Ted Faticoni came up with a counterexample with A a torsion-free abelian group of infinite rank ($W = Z$) and $R = Z[x]_{(x,2)}$, a reasonably nice integral domain.

If time permits, I'll give some examples, conjectures, and vague musings on the faithful vs fully faithful problem.