STU'S PUZZLE CORNER

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Fermat-like problems

We're all familiar with *Pythagorean triples*: triples of natural numbers (a, b, c) satisfying the equation of the Pythagorean theorem, $a^2 + b^2 = c^2$. Some commonly known Pythagorean triples are $(3, 4, 5)$, $(5, 12, 13)$ and $(8, 15, 17)$. It is obvious that there are infinitely many Pythagorean triples, if only because whenever you have one, say (a, b, c), you get an infinite family of proportional Pythagorean triples by taking $(2a, 2b, 2c)$, $(3a, 3b, 3c)$, and so on. Euclid found a more interesting way to get infinitely many: for any natural number k we have $k^2 + (2k + 1) = (k + 1)^2$, so just take k for which $2k + 1$ is a perfect square.

We can improve on even that. For any three natural numbers m,n and k with $m > n$, the triple consisting of $a = 2kmn$, $b = k(m^2 - n^2)$ and $c = k(m^2 + n^2)$ works, as you can see by an easy computation. Remarkably – and you may well have seen this – that is the whole story:

Problem 1. *Every* Pythagorean triple has this form. Not only that, but it is always possible to arrange that m and n have no common factor greater than 1, and that one of them is even and the other odd.

You might want to find such representations for your favorite Pythagorean triples, for instance those mentioned above.

It is notorious that during the seventeenth century, Pierre de Fermat announced, and claimed to have proved, that the exponent 2 is the best you can do in the above discussion. *Fermat's Last Theorem* asserts that if e is any natural number greater than 2, then the equation $a^e + b^e = c^e$ has no solution in natural numbers a, b and c. By now you know that, after more than three centuries of labor by many of the greatest mathematicians in history, in the 1990s Andrew Wiles completed the proof of this (now) theorem. So now comes the next question: what happens if you vary the exponents? For instance, we have

$$1^4 + 2^3 = 3^2 \qquad 3^2 + 6^3 = 15^2 \qquad 10^2 + 5^3 = 15^2 \qquad 13^2 + 3^3 = 14^2$$

$$13^2 + 7^3 = 8^3 = 2^9 \qquad 15^2 + 4^3 = 15^2 + 2^6 = 17^2.$$

Problem 2. Do the equations $a^2 + b^2 = c^4$, $a^4 + b^4 = c^2$, and $a^2 + b^3 = c^4$ have solutions in natural numbers?

Of course, you can try any bunch of exponents that you like. Once you get hooked, you can go after the big money. Texas banker, entrepreneur and amateur mathematician Andrew Beal has offered a prize, whose current value is at least \$100,000, for a proof or disproof of the *Beal Conjecture*: If the natural numbers a,

b, c, x, y, z with x, y and z each at least 3 satisfy the equation

$$a^x + b^y = c^z$$

then a, b and c have a common factor greater than 1. For an account of this problem (written when the prize was still $5,000) see pages 1436-1437 of volume 44, number 11 of the *Notices of the American Mathematical Society*, published in December 1997. Good luck!

Please offer suggestions or solutions *via* e-mail to:

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or *via* surface mail to:

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We plan to publish the first correct solution to problem **2** in the next issue of this newsletter. Of course, should you happen to settle the Beal conjecture, we'll publish that too, but that will be the least of your rewards. (Note: As I said last year, folks haven't been submitting solutions to past problems. Let's start a new tradition with this one!)