1. True/False

(a) $\displaystyle\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^n = 5.$

Solution: False. $\displaystyle\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^n = \frac{1}{1-\frac{4}{5}} - 1 - \frac{4}{5} = \frac{16}{5}.$

(b) If $b_n > 0$ and $b_{n+1} < b_n$ then the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n b_n$ converges.

Solution: False. Let $b_n = 1 + \frac{1}{n}$. Clearly $b_n > 0$ and $b_{n+1} < b_n$ however

$$\lim_{n\to\infty}(-1)^n b_n \neq 0$$

and so the series diverges by the test for divergence.

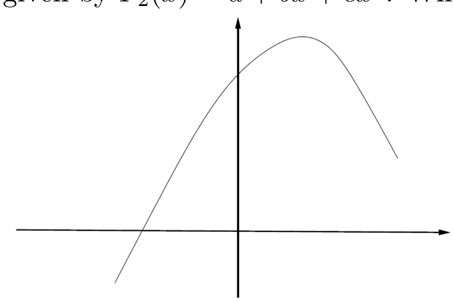
(c) $y = x^2 + x$ is a solution to $y' = 2(y - x^2) + 1$.

Solution: True. $y' = 2x + 1 = 2(x^2 + x - x^2) + 1 = 2(y - x^2) + 1$.

(d) If $\displaystyle\sum_{k=1}^{\infty} a_k x^k$ converges for $x = 1$ and $x = 2$ then it converges for $x = -2$.

Solution: False. Consider the series $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{x^k}{k2^k}$.

2. The function $f(x)$, whose graph is shown, has the Taylor polynomial of degree 2 about $x = 0$ given by $P_2(x) = a + bx + cx^2$. What can you say about the signs of a, b and c?



Solution: By Taylor's Theorem we have that $a = f(0)$, $b = f'(0)$ and $c = \frac{1}{2}f''(0)$. Since the function is positive, increasing and concave-down at $x = 0$, we must have that a and b are positive while c is negative.

3. Find the slope of the line tangent to the parametric curve $x = t\cos(t)$, $y = 3t + t^5$ when $t = 0$.

Solution: We have that (provided $\frac{dx}{dt} \neq 0$)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3 + 5t^4}{\cos(t) - t\sin(t)}.$$

When $t = 0$ we get that $\frac{dy}{dx} = 3$.

4. Convert the polar coordinates $(5, \frac{3\pi}{4})$ into Cartesian coordinates.

Solution: We have that

$$x = r\cos(\theta) = 5\cos\left(\frac{3\pi}{4}\right) = -\frac{5\sqrt{2}}{2}, \text{ and that}$$

$$y = r\sin(\theta) = 5\sin\left(\frac{3\pi}{4}\right) = \frac{5\sqrt{2}}{2}.$$

5. Find a power series that represents the function $f(x) = \frac{1}{1-x}$ and give its radius of convergence.

Solution: We have that the geometric series $\sum_{n=0}^{\infty} a^n$ converges to $\frac{1}{1-a}$ provided that $|a| < 1$ and diverges otherwise. Thus,

$$f(x) = \sum_{n=0}^{\infty} x^n$$

with radius of convergence $R = 1$.

6. A tank initially contains 100 gallons of brine in which 50 lb of salt are dissolved. A brine solution containing 2 lb/gal of salt runs into the tank at a rate of 5 gal/min. The misture is kept uniform by stirring and flows out of the tank at the same rate of 5 gal/min.

(a) If $y(t)$ is the amount of salt in the tank after t minutes, write down the initial value problem describing the mixing process.

Solution:

$$\frac{dy}{dt} = \frac{2 \text{ lb}}{\text{gal}} \cdot \frac{5 \text{ gal}}{\text{min}} - \frac{y\ lb}{100 \text{ gal}} \cdot \frac{5 \text{ gal}}{\text{min}} = 10 - \frac{y}{20}.$$

(b) Find $y(t)$, the amount of salt in the tank after t minutes.

Solution:

$$\frac{dy}{dt} = 10 - \frac{y}{20} =$$

$$\frac{1}{10 - \frac{y}{20}} \frac{dy}{dt} = 1$$

$$\int \frac{1}{10 - \frac{y}{20}} \frac{dy}{dt} dt = \int \frac{1}{10 - \frac{y}{20}} dy = \int 1 dt$$

$$-20 \ln \left| 10 - \frac{y}{20} \right| = t + C$$

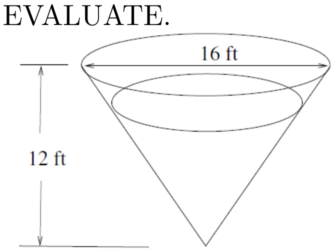
$$\left| 10 - \frac{y}{20} \right| = Ae^{-\frac{t}{20}}$$

$$y = 200 - Ae^{-\frac{t}{20}}.$$

Applying initial conditions, $y(0) = 50$, we obtain $A = 150$. Thus,

$$y(t) = 200 - 150e^{-\frac{t}{20}}.$$

7. A conical tank (shown below) has a height of 12 feet and the diameter of the top is 16 feet. It is filled to within 2 feet of the top with olive oil weighing 57 lb/ft^3. How much work does it take to pump the oil to the rim of the tank? Give your answer as a definite integral. DO NOT EVALUATE.



Solution: Divide the region into horizontally cut wafers of thickness Δx, where x is the variable denoting distance measured from the bottom of the tank. Fix $x \leq 10$. The radius of the circular wafer at level x is given by similar triangles as $r_x = \frac{2}{3}x$. The volume of this cut is given by

$$A\Delta x = \pi r_x^2 \Delta x = \frac{4}{9}\pi x^2 \Delta x,$$

and so the weight of the olive oil contained within this region is

$$57 \cdot \frac{4}{9}\pi x^2 \Delta x = \frac{76}{3}\pi x^2 \Delta x.$$

The olive oil within this region must travel $12 - x$ feet to get to the top of the tank, and so the work required to bring this olive oil to the top of the tank is $\frac{76}{3}(12 - x)\pi x^2 \Delta x$. Letting

$\Delta x \to 0$ and summing over all slices one obtains the following integral:

$$W = \frac{76}{3}\pi \int_0^{10} x^2(12-x)dx.$$

If instead, one wishes to forumate the problem with respect to the distance measured from the top of the tank y, then one obtains an equivalent integral:

$$W = \frac{76}{3}\pi \int_2^{12} y(12-y)^2 dy.$$

Note: It is obvious that these two integrals are the same since making the substitution $y = 12 - x$ in the first integral will yield the second.

8. Use $\int \frac{1}{(x+1)^{4/3}}dx = -\frac{3}{(x+1)^{1/3}} + C$ to find the value of the following improper integrals, or, if an integral does not converge, say so explicitly and show this.

(a) $\int_1^{\infty} \frac{1}{(x+1)^{4/3}}dx.$

Solution:

$$\int_1^{\infty} \frac{1}{(x+1)^{4/3}}dx = \lim_{R\to\infty} \int_1^R \frac{1}{(x+1)^{4/3}}dx = \lim_{R\to\infty} -\frac{3}{(x+1)^{1/3}}\Big|_1^R$$

$$= \lim_{R\to\infty} -\frac{3}{(R+1)^{1/3}} + \frac{3}{(2)^{1/3}} = \frac{3}{\sqrt[3]{2}} < \infty,$$

and so this improper integral converges to $\frac{3}{\sqrt[3]{2}}$.

(b) $\int_{-1}^{1} \frac{1}{(x+1)^{4/3}}dx.$

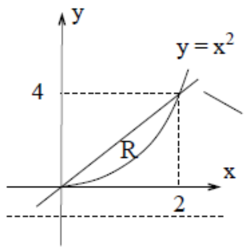
Solution:

$$\int_{-1}^{1} \frac{1}{(x+1)^{4/3}}dx = \lim_{r\to 0^+} \int_{-1+r}^{1} \frac{1}{(x+1)^{4/3}}dx = \lim_{r\to 0^+} -\frac{3}{(x+1)^{1/3}}\Big|_{-1+r}^{1}$$

$$= \lim_{r\to 0^+} -\frac{3}{(2)^{1/3}} + \frac{3}{(-1+r-1)^{1/3}} = \lim_{r\to 0^+} -\frac{3}{(2)^{1/3}} + \frac{3}{(r)^{1/3}} = \infty,$$

and so this improper integral diverges.

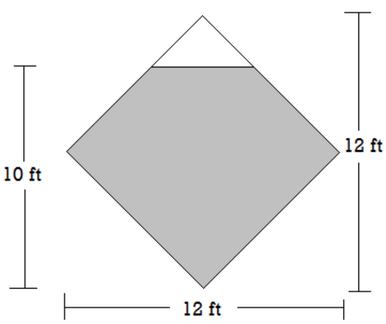
9. Consider the region R bounded by $y = 2x$ and $y = x^2$. A solid figure P has R as a base region and cross-sections perpendicular to the x-axis are squares. Express the volume of P as a definite integral. DO NOT EVALUATE.



Solution: If $A(x)$ is the function denoting cross sectional area, then for every value x, $A(x) = (x^2 - 2x)^2$. The desired integral is

$$\int_0^2 (x^2 - 2x)^2 dx.$$

10. A metal oil tank has cross-section that is a square rotated 45° is shown in the figure below. Its height and width are both 12 ft. The oil in the tank has weight-density of 57 lb/ft^3. Write an expression involving definite integral(s) that expresses the fluid force on the end of the tank when the oil is 10 feet deep. DO NOT EVALUATE.



Solution: Formulate the expression by breaking up the volume of liquid into two parts: (i) The portion above the middle of the square, where x is the distance down from the top corner, and (ii) the portion below the middle of the square where y is the distance up from the bottom corner. In this case one obtains the following expression:

$$F = 57 \int_2^6 2x(x-2)\, dx + 57 \int_0^6 2y(10-y)\, dy.$$

11. Find the centroid of the region bounded by the curves $y = \frac{x}{2}$ and $y = \sqrt{x}$.

Solution:

$$\bar{x} = \frac{M_y}{m} = \frac{\int_0^4 x(\sqrt{x} - \frac{x}{2})dx}{\int_0^4 (\sqrt{x} - \frac{x}{2})dx} = \frac{\frac{2}{5}x^{5/2} - \frac{1}{6}x^3\Big|_0^4}{\frac{2}{3}x^{3/2} - \frac{1}{4}x^2\Big|_0^4} = \frac{\frac{64}{5} - \frac{64}{6}}{\frac{16}{3} - \frac{16}{4}} = \frac{8}{5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\int_0^4 \frac{1}{2}(x - \frac{x^2}{4})dx}{\int_0^4 (\sqrt{x} - \frac{x}{2})dx} = \frac{\frac{1}{4}x^2 - \frac{1}{24}x^3\Big|_0^4}{\frac{2}{3}x^{3/2} - \frac{1}{4}x^2\Big|_0^4} = \frac{4 - \frac{8}{3}}{\frac{16}{3} - \frac{16}{4}} = 1.$$

12. Use a definite integral to express the length of the curve given by $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le \frac{\pi}{2}$.

Solution:

$$\frac{dx}{dt} = -3\cos^2(t)\sin(t)$$

$$\frac{dy}{dt} = 3\sin^2(t)\cos(t)$$

and so

$$S = \int_0^{\pi/2} \sqrt{(-3\cos^2(t)\sin(t))^2 + \left(3\sin^2(t)\cos(t)\right)^2} dt = 3\int_0^{\pi/2} \cos(t)\sin(t) dt$$

$$= \frac{3}{2}\sin^2(t)\Big|_0^{\pi/2} = \frac{3}{2}.$$

13. Write a definite integral that gives the volume of the solid of revolution formed by revolving the region bounded by the graph of $x = e^{-y^2}$ and the y-axis between $y = 0$ and $y = 1$, about the x-axis.

Solution: By method of cylindrical shells:

$$\int_0^1 (e^{-y^2} - e^{-1})2\pi y \, dy.$$

By method of discs:

$$-\pi \int_{e^{-1}}^1 \ln(x) \, dx.$$

The answer for both methods is $\pi[1 - 2e^{-1}]$.

14. Write a definite integral that gives the volume of the solid of revolution formed by revolving the region bounded by $y = x^3 + x + 1$, $y = 1$ and $x = 1$ about the line $x = 2$.

Solution: Using method of cylindrical shells:

$$V = 2\pi \int_0^1 (2-x)(x^3 + x + 1)dx = \frac{44\pi}{15}.$$

15. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{2^n}{n}(x-3)^n$.

Solution: Applying the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}}{n+1}(x-3)^{n+1}}{\frac{2^n}{n}(x-3)^n}\right| = \lim_{n\to\infty} \frac{n}{n+1} 2|x-3| = 2|x-3| < 1$$

$$|x-3| < \frac{1}{2}.$$

Thus, the power series converges absolutely for all $x \in (\frac{5}{2}, \frac{7}{2})$. For $x = \frac{5}{2}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{5}{2} - 3\right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n} \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

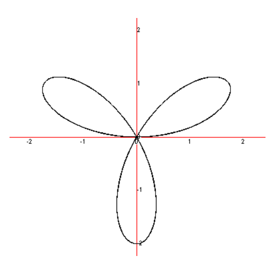
which converges. For $x = \frac{7}{2}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{7}{2} - 3\right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges. Thus, the interval of convergence is $[\frac{5}{2}, \frac{7}{2})$.

16. Sketch the graph of $r = 2\sin(3\theta)$. Give the coordinates of the points at which the curve crosses the axes. Write a definite integral that represents the area inside the region enclosed by one loop. DO NOT EVALUATE.

Solution: Below is a graph of the function:



This function is periodic with period π, and so we take $0 \leq \theta \leq \pi$ to be the domain. If $x = 0$, then $2\sin(3\theta)\cos(\theta) = 0$, implying that $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$ or $\frac{\pi}{2}$. If $y = 0$ then $2\sin(3\theta)\sin(\theta) = 0$, implying that $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$ or π. For $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$ and π we have that $\sin(3\theta) = 0$, giving us that $r = 0$. Thus, each of these values of θ correspond to the same point $x = 0$, $y = 0$. For $\theta = \frac{\pi}{2}$, $r = -1$ which gives us that this value of θ corresponds to the point $x = 0$, $y = -2$. These are the only points where the curve crosses a principal axis.

The area inside of the top right loop is given by

$$\frac{1}{2}\int_0^{\frac{\pi}{3}} r^2 \, d\theta = 2\int_0^{\frac{\pi}{3}} \sin^2(3\theta) \, d\theta.$$

17. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ converges or diverges.

Solution: Applying the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^2}{3^{n+1}}}{\frac{n^2}{3^n}}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{3n^2} = \frac{1}{3} < 1.$$

The series is absolutely convergent, and thus convergent.

18. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^4+2}}$ converges absolutely, converges conditionally or diverges.

Solution: Let $f(x) = \dfrac{x}{\sqrt{x^4+2}}$, then

$$f'(x) = \frac{\sqrt{x^4+2} - \frac{2x^4}{\sqrt{x^4+2}}}{x^4+2} = \frac{2-x^4}{(x^4+2)^{3/2}}$$

which is negative for all $x > \sqrt[4]{2}$. Thus, f is decreasing for all such x, and the sequence $\frac{n}{\sqrt{n^4+2}}$ is decreasing. Clearly this sequence is positive. We have that

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^4+2}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\sqrt{1+\frac{2}{n^4}}} = 0.$$

Thus, the given alternating series is convergent by the alternating series test. We can show that the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+2}}$ is divergent. We have that

$$\lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^4+2}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{\sqrt{n^4+2}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{2}{n^4}}} = 1.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is the well-known divergent harmonic series. By the limit comparison test the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+2}}$ is divergent. Thus, the original series is conditionally convergent.

19. Use the integral test to determine whether the series $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$ converges or diverges.

Solution: The function $f(x) = \frac{\ln(x)}{x^2}$ is positive for $x \geq 2$. We have that $f'(x) = \frac{x - 2x\ln(x)}{x^4}$ is negative for $x > 2 > \sqrt{e}$, and so the function is decreasing. It is clear by L'Hospital's Rule that $\lim_{x\to\infty} f(x) = 0$, and so $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$ is suited for the integral test. We apply integration by parts:

$$\int_2^{\infty} \frac{\ln(x)}{x^2} dx = \lim_{R\to\infty} \int_2^R \frac{\ln(x)}{x^2} dx = \lim_{R\to\infty} \left[-\frac{1}{x}\ln(x)\Big|_2^R + \int_2^R \frac{1}{x^2} dx \right]$$

$$\lim_{R\to\infty} \left[-\frac{1}{x}\ln(x) - \frac{1}{x}\Big|_2^R \right] = \frac{1+\ln(2)}{2} < \infty.$$

Thus, the given series converges by the integral test.