Fall 2007 Exam 2 Solutions

1 (a) False.

$f(x) = x$ and $g(x) = x - 1$ are increasing everywhere. $fg = x^2 - x$

$$\frac{d}{dx}(fg) = 2x - 1$$

fg is decreasing for $x < \frac{1}{2}$.

(b) False.

$\frac{1}{10}$ is a constant. Thus, $\ln(\frac{1}{10})$ is a constant, so $\frac{d}{dx}(\ln(\frac{1}{10})) = 0$.

(c) False.

Consider $f(x) = |x|$. The absolute minimum of $f(x)$ occurs at $x = 0$. $f'(x)$ is not defined at $x = 0$.

(d) False. In order for a point of inflection of f to be an extremum of f', f''' would have to be 0 there. This may not happen. Consider $f(x) = x^4 - 4x^3 + 6x^2$.

(e) False.

Let $f(x) = \begin{cases} 0 & \text{for } x = 0 \\ \frac{1}{x} & \text{for } x \neq 0 \end{cases}$

$f(x)$ has no global max on $[0, 1]$, but it is defined for all values of x in $[0, 1]$.

(2) (a) $f'(x) = x^3 - 6x^2 + 9x$

(b) $x^3 - 6x^2 + 9x = x(x^2 - 6x + 9) = x(x-3)^2$.

Thus $x = 0$ and $x = 3$ are critical points of $f(x)$.

[See below for classification].

(c) $f'(x) \geqslant 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$.

(an easy way of seeing this is to note that $(x-3)^2$ is always non-negative.)

Thus f is increasing on $(0,3) \cup (3, \infty)$ and decreasing on $(-\infty, 0)$.

(d) Keep going for the answer to this.

(e) $f''(x) = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

$= 3(x-3)(x-1)$

From this it is clear that

$f''(x) > 0$ on $(-\infty, 1) \cup (3, \infty)$ and

$f''(x) < 0$ on $(1, 3)$.

Since $f''(0) = 9 > 0$, $x = 0$ is a local minimum of $f(x)$.

Since $f''(3) = 0$, $x = 3$ is a point of inflection of $f(x)$.

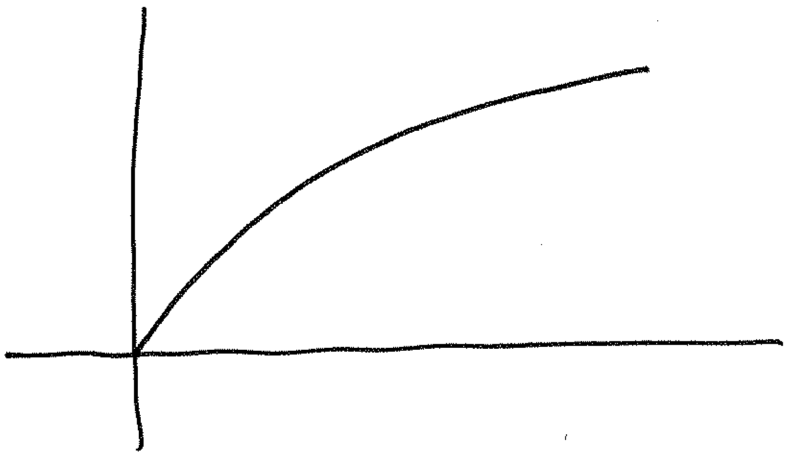
Since $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = \infty$, f has no global maximum.

Since $x = 0$ is the only local minimum, it is the global minimum as well.

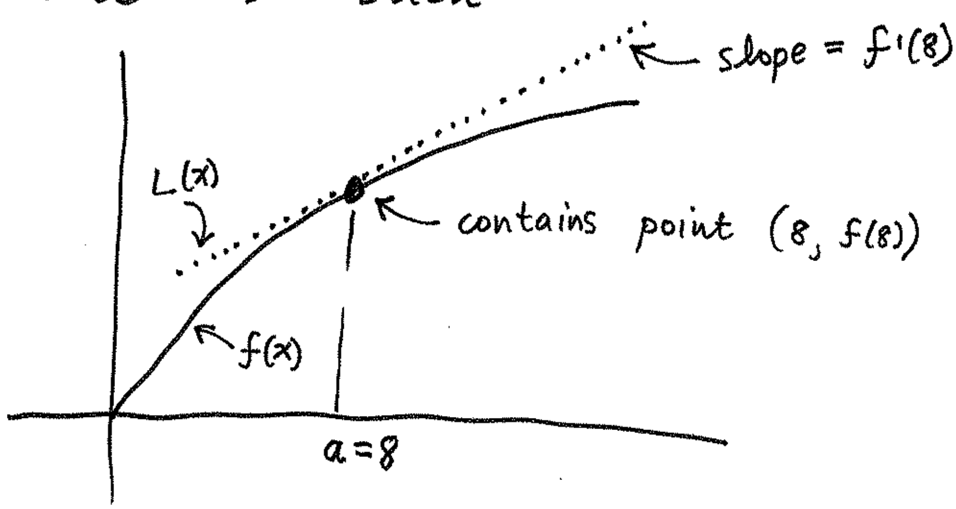
f is concave up on $(-\infty, 1) \cup (3, \infty)$ and concave down on $(1, 3)$.

③ ⓐ

$f(x) = x^{2/3}$ looks roughly like this:



We will approximate $f(x)$ with a tangent line as such:



Point-Slope formula:

$$L - f(8) = f'(8)(x-8) \longrightarrow L(x) = f(8) + f'(8)(x-8).$$

$$f'(x) = \frac{2}{3}x^{-1/3} \longrightarrow f'(8) = \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3}$$

$f(8) = 4$ Thus $\boxed{L(x) = 4 + \frac{1}{3}(x-8)}$.

(b) $L(7.98) = 4 + \frac{1}{3}(7.98-8)$

$= 4 + \frac{1}{3}(-0.02) \cong 3.99\overline{3}$.

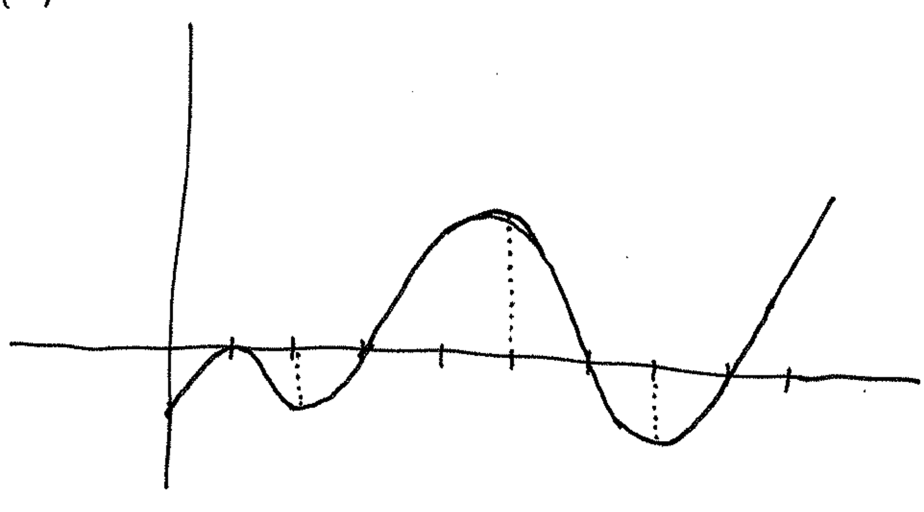
Thus $(7.98)^{2/3} \cong 3.99\overline{3}$.

(c) Since $L(x) \geq f(x)$ this is an overestimate.

(4) $\lim\limits_{x\to 0} \frac{e^{3x}-3x-1}{x^2} = \lim\limits_{x\to 0} \frac{3e^{3x}-3}{2x}$

$= \lim\limits_{x\to 0} \frac{9e^{3x}}{2} = \frac{9}{2}$.

(5) $f'(x)$:



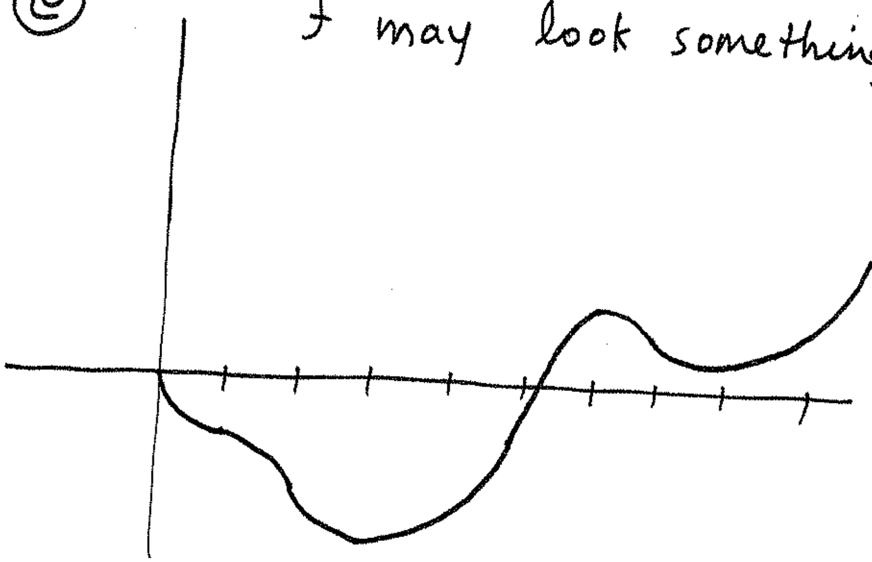
(a) $f(x)$ is increasing on $(3,6) \cup (8,\infty)$.

(b) $f(x)$ has a local max at $x=6$ and local minima at $x=3$, $x=8$.

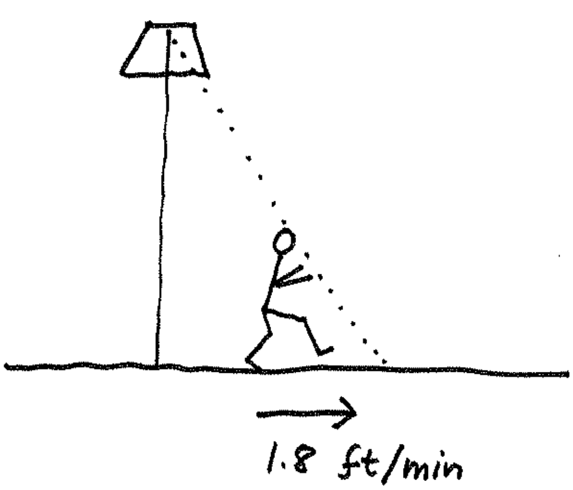
(c) f is concave up when $f'' > 0$ i.e. when f' is increasing. Thus, f is concave up on $(0,1) \cup (2,5) \cup (7,9)$.

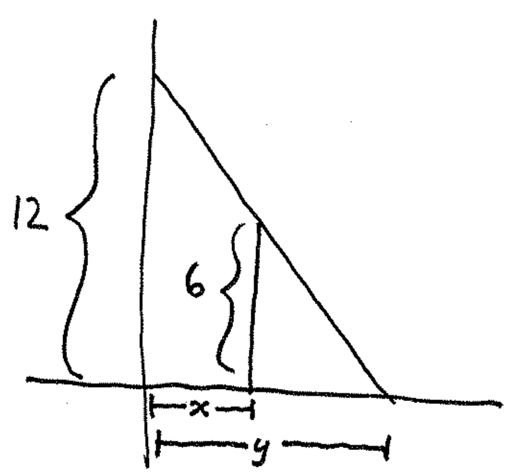
(d) Points of inflection occur when $f''(x) = 0$ (i.e. at the critical points of f').
$x = 1, 2, 5, 7$.

(e) f may look something like this:



(6)





We know that

$$\frac{dx}{dt} = 1.8 .$$

We wish to find $\frac{dy}{dt}$. We know that the large and small triangles are similar, so $\frac{y}{12} = \frac{y-x}{6}$.

Thus, $\frac{x}{6} = \frac{y}{12}$, so $2x = y$.

Differentiating both sides, and using the chain Rule:

$$\frac{d}{dt}(2x) = 2\frac{dx}{dt} = \frac{d}{dt}(y) = \frac{dy}{dt}.$$

Thus, $\frac{dy}{dt} = 2\frac{dx}{dt} = 3.6$ ft/min.

(Note: This is very slow.).

7. $$\frac{dP}{dt} = kP \longrightarrow P(t) = P(0)e^{kt}.$$

a. $P(0) = 300 \qquad P(2) = 420$

$$= 300e^{2k}$$

$$\frac{420}{300} = \frac{7}{5} = e^{2k} \longrightarrow k = \frac{\ln(7/5)}{2}.$$

$$P(t) = 300e^{t\ln(7/5)/2} = \boxed{300\left(\frac{7}{5}\right)^{t/2}}$$

(b)

$$\frac{dP}{dt} = kP \longrightarrow \text{At } t=4:$$

$$\boxed{\frac{dP}{dt} = \frac{\ln(7/5)}{2} \cdot 300\left(\frac{7}{5}\right)^2}$$

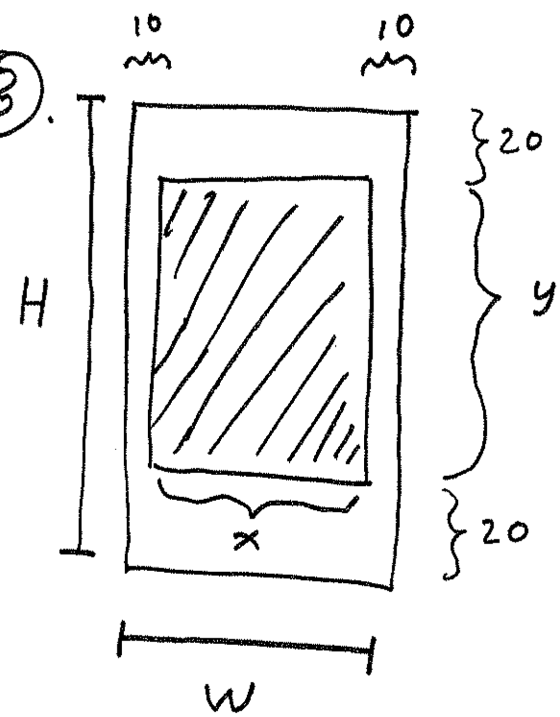
(c) Suppose $P(T) = 15{,}000$.

$$300\left(\frac{7}{5}\right)^{T/2} = 15{,}000$$

$$\left(\frac{7}{5}\right)^{T/2} = 50$$

$$\frac{T}{2}\ln\left(\frac{7}{5}\right) = \ln(50) \longrightarrow \boxed{T = \frac{2\ln(50)}{\ln(7/5)}}.$$

(8).



$$xy = 20000 \rightarrow y = \frac{20000}{x}$$

$$W = x + 20$$

$$H = y + 40 = \frac{20000}{x} + 40$$

$$A(x) = W \cdot H = (x+20)\left(\frac{20000}{x} + 40\right)$$

$$= 20000 + 40x + \frac{400000}{x} + 800$$

$$\frac{d(A)}{dx} = 40 - \frac{40{,}000}{x^2}$$

$A(x)$ has a critical point if

$$40 - \frac{40000}{x^2} = 0$$

$$x^2 = 1000$$

$$x = 10\sqrt{10}.$$

$$\frac{d^2A}{dx^2} = \frac{80000}{x^3}.$$ Since $\frac{d^2A}{dx^2} > 0$ for $x = 10\sqrt{10}$,

this is a local minimum.

$A(x)$ is defined for $x > 0$. We must check $x = 0$:

$\lim_{x \to 0^+} A(x) = \infty$ and $\lim_{x \to \infty} A(x) = \infty$ ~~fo~~ so

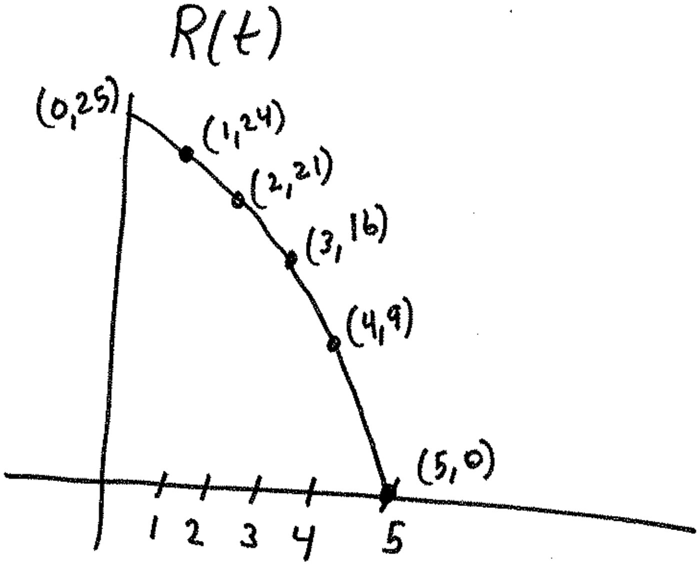
$A(x)$ has a minimum for $x = 10\sqrt{10}$.

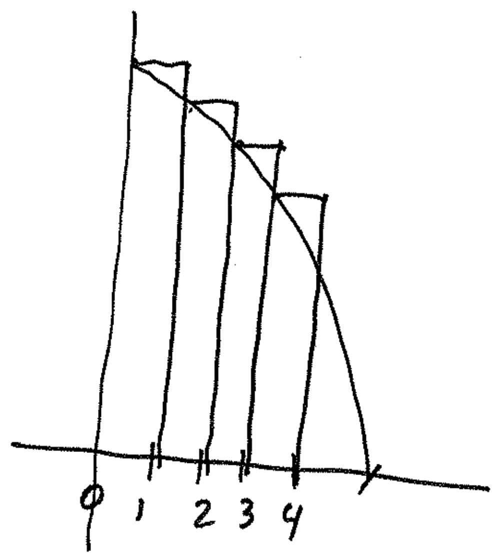
Thus, the optimal dimensions of the lot are

$$W = 10\sqrt{10} + 20$$

$$H = \frac{20000}{10\sqrt{10}} + 40 = 200\sqrt{10} + 40.$$

9





$$\int_0^4 R(t)\,dt \overset{?}{\cong} 25\times 1 + 24\times 1 + 21\times 1 + 16\times 1$$

$$= 86$$

Approximately 86 L of cider flowed out of the tank in 4 minutes.

10 a

$V(0) = 112$ $\quad H(0) = 128$ $\quad A(t) = -32$

$\frac{dH}{dt} = V(t)$ $\quad \frac{dV}{dt} = A(t)$

$V(t) = -32t + C_1$. Since $V(0) = 112$, $C_1 = 112$.

$\boxed{V(t) = -32t + 112}$

$H(t) = -16t^2 + 112t + C_2$. Since $H(0) = 128$, $C_2 = 128$.

$\boxed{H(t) = -16t^2 + 112t + 128}$

(b) $\frac{dH}{dt} = 0 \longrightarrow v(t) = 0 \longrightarrow -32t + 112 = 0$

$\longrightarrow t = \frac{112}{32} = 3.5$

$A(t) < 0$ so $t = 3.5$ gives a maximum.

$H(3.5) = -16(3.5)^2 + 112(3.5) + 128 = 324.$