1. True of False:

   (a) If the power series $\sum C_n(x+1)^n$ diverges at $x = 1$ a converges at $x = -3$ then the radius of convergence must be equal to 2.

   Solution: True. We have that the given series can be rewritten as $\sum C_n(x-(-1))^n$, which indicates that it is centered at $x = -1$. Recall that the radius of convergence is a number $R \geq 0$ (possibly infinite), such that the series converges if $|x-(-1)| < R$, diverges when $|x-(-1)| > R$, and may either converge or diverge when $|x-(-1)| = R$. Since the series converges when $x = -3$, we have that R is nonzero, and $R \geq 2$. Since the series diverges when $x = 1$ we have that R is finite, and $R \leq 2$. Thus, $R = 2$.

   (b) The Taylor series $\displaystyle\sum_{i=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^n$ converges to e^{-x} for all x.

   Solution: False. When $x = 0$ this series converges to 0. We have that

   $$e^{-0} = e^0 = 1 \neq 0$$

   .

   (c) The sequence $a_n = (1+\frac{1}{n})^{2n}$ converges to e.

   Solution: False. Recall that $(1+\frac{1}{n})^n \to e$ as $n \to \infty$. Since $(1+\frac{1}{n})^{2n} = ((1+\frac{1}{n})^n)^2$, we have that the sequence will tend to e^2. To see this directly, let $f(x) = (1+\frac{1}{x})^{2x}$. We have that $\lim_{x\to\infty}(1+\frac{1}{x})^{2x}$ has the indeterminate form 1^∞. To proceed further, we write

   $$f(x) = (1+\frac{1}{x})^{2x} = e^{\ln\left((1+\frac{1}{x})^{2x}\right)} = e^{2x\ln(1+\frac{1}{x})}.$$

   We have that

   $$\lim_{x\to\infty} 2x\ln(1+\frac{1}{x}) = \lim_{x\to\infty}\frac{2\ln(1+\frac{1}{x})}{\frac{1}{x}}.$$

   This limit is of indeterminate form $\frac{0}{0}$, to which we can directly apply L'Hospital:

   $$\lim_{x\to\infty}\frac{2\ln(1+\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{2}{1+\frac{1}{x}}\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} \lim_{x\to\infty}\frac{2}{1+\frac{1}{x}} = 2.$$

   Since e^x is continuous at $x = 2$, we have that

   $$\lim_{x\to\infty} e^{2x\ln(1+\frac{1}{x})} = e^2.$$

   Since $f(n) = a_n$, we have that $a_n \to e^2$ as $n \to \infty$.

(d) If $f(x)$ is a bounded continuous function with the properties that $f(x) > \frac{1}{x}$ for $1 \leq x < 50$, $f(50) = \frac{1}{50}$, and $f(x) < \frac{1}{x^2}$ for $x > 100$, then $\int_1^\infty f(x)\,\mathrm{d}x$ diverges.

> **Solution:** False. We have that
>
> $$\int_1^\infty f(x)\ \mathrm{d}x = \int_1^{100} f(x)\ \mathrm{d}x + \int_{100}^\infty f(x)\ \mathrm{d}x.$$
>
> Since f is bounded and continuous on $[1, 100]$ we have that $\int_1^{100} f(x)\ \mathrm{d}x$ is finite. Since $f(x) < \frac{1}{x^2}$, we have that
>
> $$\int_{100}^\infty f(x)\ \mathrm{d}x < \int_{100}^\infty \frac{1}{x^2}\ \mathrm{d}x < \infty.$$
>
> Since the given integral is the sum of two finite numbers, it is also finite. Thus, it converges.

(e) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n = 5.$

> **Solution:** True. We have that $|\frac{5}{6}| = \frac{5}{6} < 1$, and so the geometric series
>
> $$\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n = \frac{1}{1-\frac{5}{6}} = \frac{1}{\frac{1}{6}} = 6.$$
>
> We have that
>
> $$\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n = \left(\frac{5}{6}\right)^0 + \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n = 1 + \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n.$$
>
> Thus,
>
> $$\sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n = 6 - 1 = 5.$$

2. Suppose $f(3) = 4$, $f'(3) = 5$, $f''(3) = -2$, $f'''(3) = 12$ and all derivatives of f are continuous. Write down the third degree Taylor polynomial $P_3(x)$ for $f(x)$ about $x = 3$.

> **Solution:** Recall that the nth degree Taylor polynomial of f about $x = a$ is given by
>
> $$P_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)(x-a)^i}{i!}.$$
>
> Thus we have that
>
> $$P_3(x) = 4 + 5(x-3) + \frac{-2}{2}(x-3)^2 + \frac{12}{6}(x-3)^3$$

$$= 4 + 5(x-3) - (x-3)^2 + 2(x-3)^3$$

which can be rewritten, if you choose, as

$$P_3(x) = 2x^3 - 19x^2 + 65x - 74.$$

3. Convert the polar coordinates $(6, 7\pi/3)$ into Cartesian coordinates.

Solution: We have that $x = r\cos\theta = 6 \cdot \frac{1}{2} = 3$ and $y = r\sin\theta = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$. Thus, the answer is $(3, 3\sqrt{3})$.

4. Set up an integral that represents the length of the curve $x = t^4 - 3$, $y = 3\sin t + t^2$ when $2 \leq t \leq 3$.

Solution: Recall that if $x = x(t)$ and $y = y(t)$ then

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\, \mathrm{d}t.$$

We have that $\frac{\mathrm{d}x}{\mathrm{d}t} = 4t^3$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 3\cos t + 2t$. Thus,

$$L = \int_2^3 \sqrt{(3\cos t + 2t)^2 + (4t^3)^2}\, \mathrm{d}t.$$

5. Use separation of variables to solve the initial value problem $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{y-x}$ with $4y(\ln 2) = -\ln(2)$.

Solution: We have that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{y-x} = e^y e^{-x}$$

giving us that

$$e^{-y}\frac{\mathrm{d}y}{\mathrm{d}x} = e^{-x}\, \mathrm{d}x.$$

Integrating both sides with respect to x, and using the chain rule, we have that

$$\int e^{-y}\frac{\mathrm{d}y}{\mathrm{d}x}\, \mathrm{d}x = \int e^{-y}\, \mathrm{d}y = \int e^{-x}\, \mathrm{d}x$$

$$-e^{-y} = -e^{-x} + C$$

$$y(x) = -\ln(e^{-x} + C).$$

We have that

$$4y(\ln 2) = -4\ln(e^{-\ln 2} + C) = -4\ln\left(\frac{1}{2} + C\right) = -\ln(2)$$

giving us that

$$\ln\left(\frac{1}{2} + C\right) = \frac{1}{4}\ln(2) = \ln(\sqrt[4]{2}).$$

Thus, $C = \sqrt[4]{2} - \frac{1}{2}$ and $y(x) = -\ln(e^{-x} + \sqrt[4]{2} - \frac{1}{2})$.

6. The walls of a storage tank are obtained by rotating the curve $y = 2x^{3/2}$ (for $0 \le x \le 4$) around the y-axis. (Units are in feet.) The tank is filled to the top with water.

   (a) Write an integral (do not evaluate) giving the volume of water contained in the tank when it is full.

   Solution: Method of disks: Imagine slicing the tank perpendicular to the y-axis. The cross sections are disks. At some height y, the radius of the disk is given by the value of x such that $y = 2x^{3/2}$. This gives us that $x = \left(\frac{y}{2}\right)^{2/3}$. The cross-sectional area, then, is

   $$A(y) = \pi x^2 = \pi\left(\frac{y}{2}\right)^{4/3}.$$

   When $x = 0$, $y = 0$ and when $x = 0$, $y = 16$ giving us that the desired integral is

   $$\int_0^{16} \pi\left(\frac{y}{2}\right)^{4/3}\,\mathrm{d}y.$$

   Alternatively, one could use method of shells: We have that the height of the shells, at some value x is given as $16 - 2x^{3/2}$. Thus, the desired integral is

   $$\int_0^4 2\pi x(16 - 2x^{3/2})\,\mathrm{d}x.$$

   Both of these methods yield the same answer: $\dfrac{768\pi}{7}$.

   (b) Water is pumped out over the top of the tank at the rate of $8 - 0.25t$ ft^3/min, after pumping for t minutes. Write an integral (do not evaluate) that gives the work done in pumping all the water from the tank.

   Solution: Slicing the tank perpendicular to the y axis, we get that for slices with thickness Δy, the volume of the slice is given as

   $$V_y = \pi(x^2)\Delta y = \pi\left(\frac{y}{2}\right)^{4/3}\Delta y.$$

Since the units are in feet, we have that the weight density of the water is given by $\delta = 65$ lb/ft^3. Thus, the weight of the slice at level y is given by

$$\delta\pi\left(\frac{y}{2}\right)^{4/3}\Delta y$$

and the work done in pumping the water at this level to the top of the tank is given by

$$\delta\pi\left(\frac{y}{2}\right)^{4/3}\Delta y\cdot(16-y).$$

Adding up the work contributions from each slice, and letting $\Delta y \to 0$ we have that the total work done is

$$\int_0^{16}\delta\pi\left(\frac{y}{2}\right)^{4/3}(16-y)\,\mathrm{d}y.$$

Notice that this quantity is independent of the rate at which the water is pumped from the tank.

7. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{2^n}(x-3)^n$.

Solution: As with each power series, to find the interval of convergence we first apply the ratio test. We have that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+1)^3}{2^{(n+1)}}(x-3)^{(n+1)}}{\frac{n^3}{2^n}(x-3)^n}\right| = \frac{(n+1)^3}{n^3}\cdot\frac{2^n}{2^{(n+1)}}\cdot\left|\frac{(x-3)^{(n+1)}}{(x-3)^n}\right|$$

$$= \left(\frac{n+1}{n}\right)^3\cdot\frac{1}{2}\cdot|x-3| \longrightarrow \frac{|x-3|}{2} \quad \text{as} \quad n\to\infty.$$

We have that the series absolutely converges (and thus converges) when $\dfrac{|x-3|}{2} < 1$. In this case,

$$-1 < \frac{x-3}{2} < 1$$
$$-2 < x-3 < 2$$
$$1 < x < 5.$$

When $x = 1$, the series becomes

$$\sum_{n=1}^{\infty}\frac{n^3}{2^n}(-2)^n = \sum_{n=1}^{\infty}n^3(-1)^n$$

which diverges by the test for divergence. When $x = 5$, the series becomes

$$\sum_{n=1}^{\infty}\frac{n^3}{2^n}(2)^n = \sum_{n=1}^{\infty}n^3$$

which also diverges by the test for divergence. Thus, the interval of convergence for the given power series is $(1, 5)$.

8. Construct the Taylor Polynomial of degree 3 for $f(x) = \arctan(x)$ about $x = 0$.

Solution: We have that $f(0) = 0$. We have that

$$f'(x) = \frac{1}{1+x^2} \qquad \text{giving us that} \qquad f'(0) = 1.$$

We have that

$$f''(x) = \frac{-2x}{(1+x^2)^2} \qquad \text{giving us that} \qquad f''(0) = 0.$$

We have that

$$f'''(x) = \frac{(1+x^2)^2(-2) - (-2x)(2(1+x^2)(2x))}{(1+x^2)^4} \qquad \text{giving us that} \qquad f'''(0) = -2.$$

Recalling that

$$P_n(x) = \sum_{i=0}^{n} \frac{f^{(n)}(a)(x-a)^i}{i!}$$

we have that about $a = 0$:

$$P_3(x) = x - \frac{x^3}{3}.$$

9. Find the value of the following improper integrals, or, if an integral does not converge, say so explicitly and show this.

   (a) $\displaystyle\int_0^3 \frac{1}{(x-1)^{2/3}}\,\mathrm{d}x$.

Solution: We have that the integrand is undefined exactly when $x = 1$, so we express this improper integral in the following way:

$$\begin{aligned}
\int_0^3 \frac{1}{(x-1)^{2/3}}\,\mathrm{d}x &= \lim_{r\to 0^+}\left[\int_0^{1-r} \frac{1}{(x-1)^{2/3}}\,\mathrm{d}x\right] + \lim_{r\to 0^+}\left[\int_{1+r}^{3} \frac{1}{(x-1)^{2/3}}\,\mathrm{d}x\right] \\
&= \lim_{r\to 0^+}\left[3(x-1)^{1/3}\Big|_0^{1-r}\right] + \lim_{r\to 0^+}\left[3(x-1)^{1/3}\Big|_{1+r}^{3}\right] \\
&= \lim_{r\to 0^+}\left[3(-r)^{1/3} - 3(-1)^{1/3}\right] + \lim_{r\to 0^+}\left[3(2)^{1/3} - 3(r)^{1/3}\right] \\
&= 3 + 3\sqrt[3]{2}.
\end{aligned}$$

(b) $\displaystyle\int_1^\infty \frac{\ln(x)x}{x^2}\,\mathrm{d}x$.

Solution: We have that this integral simplifies to

$$\int_1^\infty \frac{\ln(x)}{x}\,\mathrm{d}x.$$

Letting $u = \ln(x)$ we have that $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ giving us that this integral becomes

$$\int_0^\infty u\,\mathrm{d}u = \lim_{R\to\infty}\int_0^R u\,\mathrm{d}u = \lim_{R\to\infty}\frac{u^2}{2}\Big|_0^R = \lim_{R\to\infty}\frac{R^2}{2} = \infty.$$

Thus, this improper integral diverges.

10. A diligent student has a slow leak in her bike tire, but has been too busy studying for exams to fix it. Assume that the pressure in the tire decreases at a rate proportional to the difference between the atmospheric pressure (15 lbs) and the tire pressure. On Monday at 6:00 PM she pumped the tire to a pressure of 85 lbs. By 6:00 PM Tuesday it was down to 75 lbs. Let $P(t)$ denote the tire pressure t days from Monday at 6:00 PM.

(a) Write down the initial value problem whose solution is $P(t)$.

Solution:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k(P-15) \qquad \text{where} \qquad P(0) = 85.$$

(b) Solve the differential equation and use the given initial conditions to find an expression for $P(t)$.

Solution: We proceed using separation of variables:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k(P-15)$$

$$\frac{1}{P-15}\frac{\mathrm{d}P}{\mathrm{d}t} = k$$

$$\int \frac{1}{P-15}\frac{\mathrm{d}P}{\mathrm{d}t}\,\mathrm{d}t = \int \frac{1}{P-15}\,\mathrm{d}P = \int k\,\mathrm{d}t$$

$$\ln|P-15| = kt + C$$

$$|P-15| = e^{kt+C} = e^{kt}e^C = Ae^{kt}$$

When $t = 0$ we have that $P = 85$ and so $P - 15 = 85 - 15 = 70 > 0$. If $P - 15$ were to become negative at some point, there must be a time when $P - 15 = 0$ (this is true because $P - 15$ is continuous, i.e., it can't go from positive to negative without crossing the x-axis). At this point, we would have that $|P - 15| = 0 = Ae^{kt}$ for some value of t. Since Ae^{kt} is never zero, this cannot happen. This means that $P - 15$ will stay positive for all values of t, and we have that $|P - 15| = P - 15$. Thus we have that

$$P - 15 = Ae^{kt}$$

$$P(t) = 15 + Ae^{kt}.$$

Using the initial condition of $P(0) = 85$ we have that

$$85 = 15 + A \qquad \text{giving us that} \qquad A = 70.$$

Using the condition that $P(1) = 75$ we have that

$$75 = 15 + 70e^{k}$$

$$\frac{60}{70} = e^{k}$$

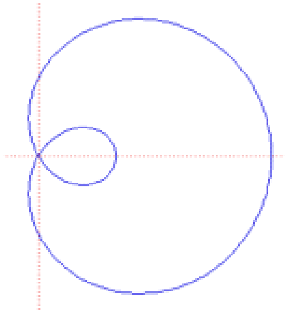
$$k = \ln\left(\frac{6}{7}\right).$$

Thus

$$P(t) = 15 + 70e^{\ln(6/7)t} = 15 + 70\left(\frac{6}{7}\right)^{t}.$$

11\. (a) Sketch the graph of $r = 1 + \sqrt{2}\cos\theta$ and give the coordinates of the points at which the curve crosses the coordinate axes.

Solution: The graph looks like the following image:



The curve crosses the coordinate axes at the following points:

r	θ	x	y
$1+\sqrt{2}$	0	$1+\sqrt{2}$	0
1	$\pi/2$	0	1
0	$3\pi/4$ and $5\pi/4$	0	0
$1-\sqrt{2}$	π	$\sqrt{2}-1$	0
1	$3\pi/2$	0	-1

(b) Write a definite integral (do not evaluate) that represents the area of the region inside the inner loop of the limacon $r = 1 + \sqrt{2}\cos\theta$.

Solution: Recall that the formula for the area contained within a curve defined by the equation $r = f(\theta)$ between θ_1 and θ_2 is given by

$$A = \int_{\theta_1}^{\theta_2} \frac{(f(\theta))^2}{2}\, \mathrm{d}\theta.$$

We have that in this case, $f(\theta) = 1 + \sqrt{2}\cos\theta$. The inner loop of the limacon is traversed as θ varies from $3\pi/4$ through $5\pi/4$. Thus, the desired integral is

$$A = \int_{3\pi/4}^{5\pi/4} \frac{(1+\sqrt{2}\cos\theta)^2}{2}\, \mathrm{d}\theta.$$

12. (a) Write down the Taylor series for $f(x) = \dfrac{1}{1+x}$ about $x = 0$.

Solution: We have that

$$f(x)_{\text{about } x=a} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

First, make a table to attempt to find a pattern describing $f^{(n)}(0)$ in terms of n:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\frac{1}{1+x}$	1
1	$\frac{-1}{(1+x)^2}$	-1
2	$\frac{2(1+x)}{(1+x)^4}$	2
3	$\frac{(1+x)^4(2)-2(1+x)(4(1+x)^3)}{(1+x)^8}$	-6
4	$\frac{(1+x)^8(8(1+x)^3-32(1+x)^3)-((1+x)^4(2)-2(1+x)(4(1+x)^3))(8(1+x)^7)}{(1+x)^{16}}$	24

It appears that $f^{(n)}(0) = (-1)^n n!$. This gives us that the Taylor series for the given function is $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n$. This series should look familiar. Recall that

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n$$

is just a geometric series. (In this case, going through the whole process of calculating the Taylor coefficients is unnecessary if one remembers that this function has an expansion similar to that of $\dfrac{1}{1-x}$).

(b) Use your result from part (a) to write down the Taylor series about $x = 0$ for $\ln(x+1)$.

Solution: Since we have that

$$\int \frac{1}{1+x}\,\mathrm{d}x = \ln(1+x) + C$$

we have that

$$\ln(x+1) = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$

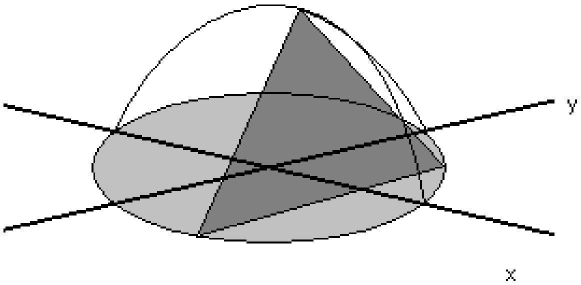
Since $\ln(1) = 0$ we have that $C = 0$ giving us that

$$\ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

is the desired expansion (recall the theorem in section 11.9 of the text).

13. A solid figure P has a base region $\{(x, y|x^2 + y^2 \leq 4)\}$ in the xy-plane. Every cross section of P by a plane perpendicular to the x-axis is an equilateral triangle. Find the volume of P.

Solution: The base of P is a circle of radius 2 lying within the xy-plane, and cross sections of P are equilateral triangles:



We will integrate with respect to x. In this case, the volume of the figure can be expressed as

$$V = \int_a^b A(x)\, \mathrm{d}x$$

where $A(x)$ is the cross sectional area of a slice made perpendicular to the x-axis, going through point x. We have that this cross section is an equilateral triangle, and the base of this triangle is twice the value of the function:

$$b = 2\sqrt{4 - x^2},$$

and that the height of this triangle is given as

$$h = \left(2\sqrt{4 - x^2}\right) \sin(\pi/3) = \left(2\sqrt{4 - x^2}\right) \left(\frac{\sqrt{3}}{2}\right).$$

Thus, we have that

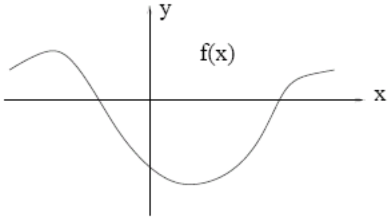
$$A(x) = \frac{1}{2} \left(2\sqrt{4 - x^2}\right)^2 \left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}(4 - x^2).$$

To find the volume then, we have

$$V = \int_{-2}^{2} \sqrt{3}(4 - x^2)\, \mathrm{d}x = \sqrt{3} \left(4x - \frac{x^3}{3}\Big|_{-2}^{2}\right)$$

$$= \frac{32\sqrt{3}}{3}.$$

14. The graph of a function $f(x)$ is shown below. Which of the following could be the Taylor polynomial approximating $f(x)$ for x near 0? More than one answer is possible. Justify your answer.



(a) $P_2(x) = 2 + 2x + 2x^2$
(b) $P_2(x) = 2 - 2x + 2x^2$
(c) $P_2(x) = 2 + 2x - 2x^2$
(d) $P_2(x) = 2 - 2x - 2x^2$
(e) $P_2(x) = -2 + 2x + 2x^2$
(f) $P_2(x) = -2 - 2x + 2x^2$
(h) $P_2(x) = -2 - 2x + 5x^2$
(i) $P_2(x) = -2 - 2x - 2x^2$
(j) $P_2(x) = -2 + 2x - 2x^2$

Solution: We have that

$$P_2(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

Since the function is negative at $x = 0$ we have that $f(0) < 0$. This eliminates equations (a), (b), (c) and (d) from the realm of possibility. Since the $f(x)$ is decreasing at $x = 0$ we have that $f'(0) < 0$. This further eliminates (e) and (j). Since $f(x)$ is concave-up at $x = 0$, we have that $f''(0) > 0$, eliminating (i). Only (f) and (h) remain. Both of these choices could be the the Taylor polynomial approximating $f(x)$ near $x = 0$.

15. Determine whether each of the following series converges absolutely, converges conditionally or diverges and then briefly explain WHY for each series.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n n e^{-n^2}$.

Solution: First, we apply the ratio test to test for absolute convergence:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)e^{-(n+1)^2}}{(-1)^n n e^{-n^2}}\right| = \frac{n+1}{n}\frac{e^{-n^2-2n-1}}{e^{-n^2}}$$

$$= \frac{n+1}{n}e^{-2n-1} \to 0 \qquad \text{as} \qquad n \to \infty.$$

Thus, this series converges absolutely.

(b) $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{\sqrt{n}}{2n^3-7}$.

Solution: If you attempt to use the ratio test for investigating absolute convergence, you will obtain no information (the ratio will tend to 1). To investigate absolute convergence, consider the series $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{2n^3-7}$ directly. When $n \geq 2$ the terms in the series are all positive. Denote the terms in this series by a_n. Consider the series $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{2n^3}$. Denote the terms in this series by b_n. This second series is in fact a p-series:

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{2n^3} = \sum_{n=2}^{\infty} \frac{1}{2n^{5/2}} = \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n}\right)^{5/2}$$

which is convergent since $5/2 > 1$. Now we apply the limit comparison test:

$$\frac{a_n}{b_n} = \frac{\frac{\sqrt{n}}{2n^3-7}}{\frac{\sqrt{n}}{2n^3}} = \frac{2n^3}{2n^3-7} \to 1 \qquad \text{as} \qquad n \to \infty.$$

Thus, the series $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{2n^3-7}$ converges by the limit comparison test. Since

$$\sum_{n=2}^{\infty} \left|(-1)^n \frac{\sqrt{n}}{2n^3-7}\right| = \sum_{n=2}^{\infty} \frac{\sqrt{n}}{2n^3-7},$$

we have that $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{\sqrt{n}}{2n^3-7}$ converges absolutely.

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \left(1+\frac{1}{n^2}\right)$.

Solution: We have that this series is alternating. Let $b_n = \left(1+\frac{1}{n^2}\right)$. We have that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right) = 1 \neq 0$$

thus, this series diverges, by the alternating series test.