NAME:  KEY (remember there may be other ways to solve the problems here)

This sample exam is just a guide to prepare for the actual exam. Questions on the actual exam may or may not be of the same type, nature, or even points. Don’t prepare only by taking this sample exam. You also need to review your class notes, homework and quizzes on WebAssign, quizzes in discussion section, and worksheets.

The exam will cover up through section 3.3 (Derivatives of trig functions). Section 2.4 is not on the exam.

Read This First!

• Please read each question carefully. Other than the question of true/false items, show all work clearly in the space provided. In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

• Answers must be clearly labeled in the spaces provided after each question. Please cross out or fully erase any work that you do not want graded. The point value of each question is indicated after its statement. No books or other references are permitted.

• Give any numerical answers in exact form, not as approximations. For example, one-third is \( \frac{1}{3} \), not .33 or .33333. And one-half of \( \pi \) is \( \frac{1}{2} \pi \), not 1.57 or 1.57079.

• Calculators are allowed but you must show all your work in order to receive credit on the problem.

Grading - For Administrative Use Only

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1. If the statement is always true, circle the printed capital T. If the statement is sometimes false, circle the printed capital F. In each case, write a careful and clear justification or a counterexample.

(a) The line $x = 1$ is a vertical asymptote of the graph of $y = \frac{x^2 - 1}{x^2 - 2x + 1}$.  

Justification:

$$\frac{x^2 - 1}{x^2 - 2x + 1} = \frac{(x-1)(x+1)}{(x-1)^2} = \frac{x+1}{x-1} \text{ as } x \to 1^+$$

\[ x = 1 \text{ is a vertical asymptote} \]

(b) The function $y(t) = 3 + 6e^{-kt}$, with $k$ a positive constant, has horizontal asymptote $y = 6$.

Justification:

Since $k > 0$, $e^{-kt} \to 0$ as $t \to \infty$ and $\lim_{t \to \infty} (3 + 6e^{-kt}) = 3 + 6(0) = 3$.

(c) A function that is not continuous at a point can not be defined at that point.

(d) If $f(x)$ is differentiable then $f(x)$ and $f(x) - 2$ have the same derivative.

Justification: Using sum rule

$$(f(x) - 2)' = f'(x) - 2' = f'(x) - 0 = f'(x)$$
2. (a) If \( f(x) = \frac{x}{x+3} \) for \( x \neq -3 \), then find its inverse function \( f^{-1}(x) \) and determine the domain of the inverse function.

\[
y = \frac{x}{x+3} \\
y(x+3) = x \\
xy - 3y = x \\
xy - x = 3y \\
x(y - 1) = 3y \\
x = \frac{3y}{y - 1} \\
f^{-1}(x) = \frac{3x}{x - 1}
\]

Domain of \( f^{-1}(x) = \frac{3x}{x - 1} \)

All \( x \) except \( x = 1 \)

(b) If \( \log_2 a = .83 \) and \( \log_2 b = .56 \), compute \( \log_2 \left( \frac{a^3}{b^2} \right) \) and give your numerical answer exactly.

\[
\log_2 \left( \frac{a^3}{b^2} \right) = \log_2 a^3 - \log_2 b^2 \\
= 3 \log_2 a - 2 \log_2 b \\
= 3(0.83) - 2(0.56) \\
= 2.49 - 1.12 \\
= \boxed{1.37}\]
3. Let $f(x) = \ln(4x+1) + 1$, $g(x) = \sqrt{3x+6}$, and $h(x) = x^3$.
   (a) Find the domain of $f(x)$.

   \[
   f(x) \text{ is defined provided } 4x + 1 > 0 \\
   4x > -1 \\
   x > -\frac{1}{4} \\
   \therefore \text{ Domain of } f(x) \text{ is } \left[-\frac{1}{4}, \infty\right)
   \]

   (b) Find the domain of $(g \circ h)(x)$

   \[
   g(h(x)) = \sqrt{3x^3 + 6} \\
   3x^3 + 6 > 0 \\
   x^3 > -\frac{6}{3} \\
   x^3 > -2 \\
   x^3 \geq -\sqrt[3]{2} \\
   \text{Domain is } \left[-\sqrt[3]{2}, \infty\right) \\
   \]

   (c) Find $f^{-1}(x)$ and its domain.

   \[
   f(x) = \ln(4x+1) + 1 \\
   y = \ln(4x+1) + 1 \\
   y - 1 = \ln(4x+1) \\
   e^{y-1} = 4x+1 \\
   x = \frac{e^{y-1} - 1}{4} \\
   \]

   \[
   f^{-1}(x) = \frac{e^{x-1} - 1}{4} \\
   \text{Domain } (-\infty, \infty)
   \]
4. Determine the following limits, using algebraic methods to simplify the expression before finding the limit. Evaluate each of the following limits. If the limit does not exist but goes to \( \infty \) or \(-\infty\), indicate so. If the limit does not exist for any other reason, write DNE with a justification.

(a) \[
\lim_{{x \to 2}} \frac{3-x}{1/3 - 1/x} = \frac{3-x}{\frac{1}{3} - \frac{1}{x}} = \frac{3-x}{\frac{x-3}{3x}} = \frac{(3-x)3x}{x-3} = -3x
\] [4]

\[
\lim_{{x \to 2}} \frac{3-x}{\sqrt{3} - \sqrt{x}} = -3(2) = -6
\]

(b) \[
\lim_{{x \to 0}} \frac{x^2 - 1}{x^2 - x} = \lim_{{x \to 0}} \frac{(x-1)(x+1)}{x(x-1)} = \lim_{{x \to 0}} \frac{x+1}{x} = 1
\] [4]

\[
\lim_{{x \to 0^+}} \frac{1}{x} = +\infty \quad \lim_{{x \to 0^-}} \frac{1}{x} = -\infty
\]

L.H.S \neq R.H.S \quad \text{Limit} \quad \lim_{{x \to 0}} \frac{x^2 - 1}{x^2 - x} \quad \text{DNE}

(c) \[
\lim_{{x \to 4}} \frac{x-4}{\sqrt{x} - 2} = \lim_{{x \to 4}} \frac{\sqrt{x} + 2}{x - 4} = \lim_{{x \to 4}} \frac{(x-4)(\sqrt{x} + 2)}{x - 4} = \sqrt{4} + 2 = 4
\] [4]
5. Let \( f(x) = \frac{2x^3 + x^2 - 7}{\sqrt{4x^6 + 9x^2 + x}} \).

(a) Compute \( \lim_{x \to \infty} f(x) \).

Multiply and divide by highest power of \( x \) in the deno. 

\[
\lim_{x \to \infty} \frac{2x^3 + x^2 - 7}{x^3} = \lim_{x \to \infty} \frac{2x^3}{x^3} - \frac{x^2}{x^3} - \frac{7}{x^3} = \sqrt{x^6} = x^3 \text{ for } x > 0
\]

(b) Compute \( \lim_{x \to -\infty} f(x) \).

When \( x \to -\infty \) we care about \( x < 0 \) and \( \sqrt{x^6} = (1x^3)^3 \) now for example \( \sqrt{-2}^6 = \sqrt{64} = 8 = (1-2y)^2 \), not \((-2)^2\), therefore 

\[
\lim_{x \to -\infty} \frac{2 - \frac{1}{x} - \frac{7}{x^3}}{\sqrt{4 + \frac{9}{x} + \frac{1}{x^5}}} = \lim_{x \to -\infty} f(x) = \frac{2}{-\sqrt{4}} = -\frac{2}{2} = \frac{1}{7}
\]

(c) What are all the horizontal asymptotes of the graph of \( y = f(x) \)?

From (a) & (b), the line \( y=k \) that could be asymptotes to \( y=f(x) \) are those with \( k = \frac{2}{7} \) and \( k = -\frac{2}{7} \) so horizontal asymptotes are \( y = \frac{2}{7} \) and \( y = -\frac{2}{7} \). 
6. Let \( f(x) = \begin{cases} 
3 - kx, & \text{if } x > 1, \\
k - x, & \text{if } x \leq 1. 
\end{cases} \)

(a) \( \lim_{x \to 1^-} f(x) = \)  

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} k - x = k - 1 
\]

(b) \( \lim_{x \to 1^+} f(x) = \)  

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1} 3 - kx = 3 - k 
\]

(c) Find the constant \( k \) so that \( f(x) \) is continuous at every point. \( k = 2 \)  

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \\
k - 1 = 3 - k \\
2k = 4 \\
k = 2 
\]
7. Find the first derivative of the following functions.

(a) \( f(x) = \sin(x) \cdot x^2 \)

\[
\frac{df}{dx} = \frac{d}{dx} (\sin(x)) \cdot x^2 + \sin(x) \cdot \frac{d}{dx} x^2 = \\
\cos(x) \cdot x^2 + \sin(x) \cdot 2x
\]

(b) \( f(x) = \frac{e^x \cdot x^2}{\cos(x)} \)

\[
\frac{df}{dx} = \frac{\cos(x) \cdot \frac{d}{dx} (e^x \cdot x^2) - e^x \cdot x^2 \cdot \frac{d}{dx} \cos(x)}{(\cos(x))^2} = \\
\frac{\cos(x) \left[ \frac{d}{dx} e^x \cdot x^2 + e^x \cdot \frac{d}{dx} x^2 \right] - e^x \cdot x^2 (-\sin(x))}{(\cos(x))^2} = \\
\frac{\cos(x) (e^x \cdot x^2 + e^x \cdot 2x) + e^x \cdot x^2 \cdot \sin(x)}{(\cos(x))^2}
\]

(c) \( f(x) = \frac{\sqrt{x^2 + 1}}{e^x \cdot x^2} \)

\[
\frac{df}{dx} = \frac{e^x \cdot x^2 \cdot \frac{d}{dx} (x^2 + 1)^{1/2} - (x^2 + 1)^{1/2} \cdot \frac{d}{dx} (e^x \cdot x^2)}{(e^x \cdot x^2)^2} = \\
\frac{e^x \cdot x^2 \cdot \frac{1}{2} (x^2 + 1) \cdot \frac{d}{dx} (x^2 + 1) - (x^2 + 1)^{1/2} \cdot (e^x \cdot x^2 + e^x \cdot 2x)}{(e^x \cdot x^2)^2}
\]

Note: \( \frac{d}{dx} (x^2 + 1)^{1/2} = \frac{1}{2} (x^2 + 1)^{-1/2} \cdot \frac{d}{dx} (x^2 + 1) \)

 uses chain rule Not on the exam!
8. Find the derivatives of the following functions. Simplify your answers (e.g., combine like powers of $x$ in a polynomial).

(a) $f(x) = 9x - 7\sqrt{x}$.
\[
f'(x) = 9 \cdot \frac{d}{dx}(x) - 7 \cdot \frac{d}{dx} x^{\frac{1}{2}} = 9 \cdot 1 - 7 \cdot \frac{1}{2} x^{-\frac{1}{2}}
\]

(b) $f(x) = x^4 e^x$.
\[
f'(x) = \frac{d}{dx} x^4 e^x + x^4 \frac{d}{dx} e^x = 4x^3 e^x + x^4 e^x = e^x x^3 (4 + x)
\]

(c) $f(x) = \frac{x^2 + x}{x^3 - 3}$.
\[
f'(x) = \frac{(x^3 - 3) \frac{d}{dx}(x^2 + x) - (x^2 + x) \frac{d}{dx}(x^3 - 3)}{(x^3 - 3)^2}
\]
\[
f'(x) = (x^2 - 3)(2x + 1) - (x^2 + x)(3x^2 - 0)
\]

(d) $f(x) = \frac{e^x}{x^2 + 1}$.
\[
f'(x) = \frac{e^x \frac{d}{dx}(x^2 + 1) - e^x \frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2} = \frac{e^x (x^2 + 1) - e^x \cdot 2x}{(x^2 + 1)^2}
\]
9. (a) Find the slope of the tangent line to the graph of \( y = (x^2 - 1)^3 \) at \((2, 27)\).

\[
y = (x^2 - 1)^3 = x^6 - 3x^4 + 3x^2 - 1
\]

\[
y' = 6x^5 - 12x^3 + 6x - 0
\]

\[
y'|_{x=2} = 6(2)^5 - 12(2)^3 + 6(2)
\]

\[
= 6(32) - 12(8) + 12 = 108.
\]

(b) Find the equation of the tangent line in part a, writing the answer as \( y = mx + b \).

\[
m = 108 \quad \text{pt } (2, 27)
\]

\[
y - 27 = 108(x - 2)
\]

\[
y = 108x - 216 + 27
\]

\[
\boxed{y = 108x - 189}
\]
10. The picture below is the graph of

\[ f(x) = \begin{cases} 
  x^3 - x, & \text{if } x \leq 1, \\
  -x^2 + ax + b, & \text{if } x > 1,
\end{cases} \]

where \( a \) and \( b \) are chosen to make the function differentiable at \( x = 1 \). Determine \( a \) and \( b \).

(Hint: the function has to be continuous at \( x = 1 \) also, since it is differentiable.)

\[
\begin{align*}
\text{Continuity} & \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \\
\lim_{x \to 1^-} x^3 - x &= \lim_{x \to 1^+} -x^2 + ax + b \\
0 &= -1 + a + b \quad \Rightarrow \quad a + b = 1
\end{align*}
\]

\[
\begin{align*}
\text{Differentiability} & \quad f'(x) = \begin{cases} 
  3x^2 - 1, & x < 1 \\
  -2x + a, & x > 1
\end{cases} \\
\lim_{x \to 1^-} 3x^2 - 1 &= \lim_{x \to 1^+} -2x + a \\
3 &= -2 + a \quad \Rightarrow \quad a = 4 \\
b &= 1 - a \quad \Rightarrow \quad b = -3
\end{align*}
\]
11. In the graph of $y = x^2/3$, which is illustrated below, find the number $c$ such that the tangent lines to the graph at the points where $x = 1$ and $x = c$ are perpendicular.

Lines with perpendicular slopes $m_1$ and $m_2$ have $m_2 = -\frac{1}{m_1}$. Now $y = \frac{x^2}{3}$

$y' = \frac{2}{3} x$ \hspace{1cm} \therefore \hspace{1cm} y'(1) = \frac{2}{3}$

$y'(c) = \frac{2}{3} c$

To be perpendicular, these two tangents lines need to have

$\frac{2}{3} c = -\frac{1}{2/3} = -\frac{3}{2}$

$C = \frac{-3}{2} \cdot \frac{2}{2} = -\frac{9}{4}$

$\boxed{C = -\frac{9}{4}}$