1. Differentiate $f(x) = x^x$.

Solution: Use logarithmic differentiation.

$$y = x^x$$
$$\ln(y) = \ln(x^x)$$
$$\ln(y) = x \ln(x)$$

$$\frac{y'}{y} = \frac{x}{x} + (1)(\ln(x))$$
$$y' = y(1 + \ln(x))$$
$$y' = x^x (1 + \ln(x)).$$

2. There are 10 bacteria in a petri dish. Twelve hours later there are 2000 bacteria in the dish. How long does it take the population to double in size? How long before there are 10000 bacteria?

Solution: Given $P(0) = 10$, $P(12) = 2000$, and $P$ is exponentially increasing; $P(t) = P(0)e^{kt} = 10e^{kt}$. Solve for $k$:

$$P(12) = 10e^{12k} = 2000$$
$$e^{12k} = \frac{2000}{10}$$
$$e^{12k} = 200$$
$$\ln(e^{12k}) = \ln(200)$$
$$12k = \ln(200)$$
$$k = \frac{\ln(200)}{12}$$

The model is

$$P(t) = 10e^{\frac{\ln(200)}{12} t}.$$ 

The population doubles in size when $P = 20$:

$$P(t) = 10e^{\frac{\ln(200)}{12} t} = 20$$
$$e^{\frac{\ln(200)}{12} t} = \frac{20}{10}$$
$$\ln(e^{\frac{\ln(200)}{12} t}) = \ln(2)$$
$$\frac{\ln(200)}{12} t = \ln(2)$$
$$t = \frac{12 \ln(2)}{\ln(200)} \approx 1.57 \text{ hours} \approx 1 \text{ hr and 34 min}.$$ 

The population reaches 10,000 bacteria when $P = 10000$:

$$P(t) = 10e^{\frac{\ln(200)}{12} t} = 10000$$
$$e^{\frac{\ln(200)}{12} t} = \frac{10000}{10}$$
$$\ln(e^{\frac{\ln(200)}{12} t}) = \ln(1000)$$
$$\frac{\ln(200)}{12} t = \ln(1000)$$
$$t = \frac{12 \ln(1000)}{\ln(200)} \approx 15.65 \text{ hours} \approx 15 \text{ hrs and 39 min}.$$
3. A cup of coffee at 180 degrees Fahrenheit is placed in a 75 degree room. When the coffee is 120 degrees, it is cooling at a rate of 3 degrees per minute. How long did it take for the coffee to get to 120 degrees?

Solution: Use Newton’s law of cooling with \( T_s = 75 \), \( T(0) = 180 \).

Recall that if \( y(t) = T(t) - T_s \), \( y'(t) = T'(t) \) and \( y(0) = T(0) - T_s \). Newton’s law of cooling says that

\[
\frac{dy}{dt} = ky,
\]

so expressed in terms of \( T \) this says

\[
\frac{dT}{dt} = k(T - T_s).
\]

We are given that \( \frac{dT}{dt} = -3 \) (negative because of cooling) when \( T = 120 \).

Plug this into the previous equation to solve for \( k \):

\[
\frac{dT}{dt} = k(T - T_s)
\]
\[
-3 = k(120 - 75)
\]
\[
-3 = \frac{45}{45} = k.
\]

So \( k = -\frac{3}{45} \approx -0.067 \).

Solve \( T(t) = 120 \) to find how long it took for the coffee to get to 120 degrees:

\[
y(t) = y(0)e^{kt}
\]
\[
T(t) - T_s = (T(0) - T_s)e^{-0.067t}
\]
\[
T(t) = T_s + (T(0) - T_s)e^{-0.067t}
\]
\[
120 = 75 + (180 - 75)e^{-0.067t}
\]
\[
45 = 105e^{-0.067t}
\]
\[
\frac{45}{105} = e^{-0.067t}
\]
\[
\ln \left( \frac{45}{105} \right) = \ln (e^{-0.067t})
\]
\[
\ln \left( \frac{45}{105} \right) = -0.067t
\]
\[
\frac{\ln \left( \frac{45}{105} \right)}{-0.067} = t
\]
\[
12.65 \approx t
\]

The coffee reaches 120 degrees after approximately 13 minutes.
4. A water trough has equilateral trapezoid cross-sections. The width of the trough is 2 feet at the top and 1 foot at the bottom. The trough is 3 feet deep and 10 feet long. Water is being pumped into the trough at a rate of 0.25 ft$^3$/s per second. How fast is the water level rising when the water is 18 inches deep?

Solution: Let $h$ be the height (or depth) of the water and $V$ be the volume of water in the trough.

Given: $\frac{dV}{dt} = 0.25$ ft$^3$/s. Unknown: $\frac{dh}{dt}$ when $h = 18$ in $= 1.5$ ft.

The volume of water in the trough is equal to the area of the cross-section of water times the length of the trough (10 ft). The area of the cross-section of water is the area of the rectangle in the middle plus the areas of the two triangles on the sides (see picture). Since the trapezoid is equilateral, the side triangles are the same, so we can just find the area of one triangle and double it. Let $b$ be the base of the side triangle.

$V = [\text{area of cross-section of water}] \times 10 \text{ ft}$

$V = [\text{area of rectangle + twice the area of one side triangle}] \times 10 \text{ ft}$

$V = 10 \left( h + \frac{1}{2}bh \right) \cdot 10$

$V = 10 \left( h + bh \right)$

Use similar triangles to solve for $b$ in terms of $h$:

\[
\frac{b}{h} = \frac{0.5}{3} \Rightarrow b = \frac{h}{6}
\]

\[
V = 10 \left( h + \frac{h}{6} \right) h
\]

$V = 10 \left( h + \frac{h^2}{6} \right)$

\[
\frac{dV}{dt} = 10 \left( \frac{dh}{dt} + \frac{2h}{6} \frac{dh}{dt} \right)
\]

\[
0.25 = 10 \left( \frac{dh}{dt} + \frac{1}{3} (1.5) \frac{dh}{dt} \right)
\]

\[
0.025 = 1.5 \frac{dh}{dt}
\]

\[
0.0167 \approx \frac{dh}{dt}
\]

The water level is rising at a rate of approximately 0.0167 ft/s.
5. An airplane cruising at an altitude of 1 mile flies over an observer on the ground at a speed of 400 mph. If the observer continually turns their head to face the plane, how fast are they rotating their head when the plane is 3 miles away?

Solution: Let \( x \) be the horizontal distance from the observer to the airplane and \( \theta \) be the angle between the observer and the plane.

Given: \( \frac{dx}{dt} = 400 \text{ mph} \)  \( \quad \) Unknown: \( \frac{d\theta}{dt} \) when \( x = 3 \text{ mi} \).

\[ \tan(\theta) = \frac{1}{x} \]
\[ \sec^2(\theta) \frac{d\theta}{dt} = -\frac{1}{x^2} \frac{dx}{dt} \]

When \( x = 3 \), the hypotenuse of the right triangle is \( \sqrt{3^2 + 1^2} = \sqrt{10} \) and so \( \frac{\text{hyp.}}{\text{adj.}} = \frac{\sqrt{10}}{3} \).

\[ \sec^2(\theta) \frac{d\theta}{dt} = -\frac{1}{x^2} \frac{dx}{dt} \]
\[ \left( \frac{\sqrt{10}}{3} \right)^2 \frac{d\theta}{dt} = -\frac{400}{3^2} \]
\[ \frac{10}{9} \frac{d\theta}{dt} = -\frac{400}{9} \]
\[ \frac{10}{9} \frac{d\theta}{dt} = -400 \]
\[ \frac{d\theta}{dt} = -40 \]

The observer is rotating his head at a rate of 40 rad/s.

6. A spherical snowball is melting at a constant rate of 3 cm\(^3\)/minute. How fast is the radius of the snowball shrinking when it is 10 cm wide?

Solution: Let \( V \) be the volume of the snowball with radius \( r \). Given \( \frac{dV}{dt} = -3 \) (melting \( \Rightarrow \) decreasing), find \( \frac{dr}{dt} \) when \( r = 5 \) (diameter = 2r). The volume of a sphere is given by \( V = \frac{4}{3} \pi r^3 \). Taking derivatives
with respect to \( t \),

\[
V = \frac{4}{3} \pi r^3
\]

\[
\frac{dV}{dt} = \frac{4\pi}{3} \cdot 3r^2 \cdot \frac{dr}{dt}
\]

\[
\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}
\]

\[
-3 = 4\pi (5)^2 \frac{dr}{dt}
\]

\[
-\frac{3\pi}{100} = \frac{dr}{dt}
\]

The radius is decreasing at a rate of approximately 0.0942 cm/min.

7. (a) Let \( f(x) = e^x + 4x^2 \). Find the linearization of \( f \) at \( a = 2 \).

\[\text{Solution:}\]

\[L(x) = f(a) + f'(a)(x - a).\]

\[f'(x) = e^x + 8x\]

\[f'(2) = e^2 + 16\]

\[L(x) = f(2) + f'(2)(x - 2)\]

\[L(x) = e^2 + 16 + (e^2 + 16)(x - 2)\]

(b) Approximate \( \ln(0.92) \) without the aid of a calculator.

\[\text{Solution: Let } f(x) = \ln(x). \text{ Approximate } \ln(0.92) \text{ by the linearization } L(x) \text{ of } f \text{ at } a = 1 \text{ (since } 0.92 \approx 1).\]

\[f'(x) = \frac{1}{x}\]

\[f'(1) = \frac{1}{1} = 1\]

\[L(x) = f(1) + f'(1)(x - 1)\]

\[L(x) = \ln(1) + 1(x - 1)\]

\[L(x) = 0 + x - 1\]

\[L(x) = x - 1.\]

Then \( \ln(0.92) \approx L(0.92) = 0.92 - 1 = -0.08. \)

8. (a) State Rolle’s Theorem.

\[\text{Solution: If } f \text{ is continuous on } [a, b], \text{ differentiable on } (a, b), \text{ and if } f(a) = f(b), \text{ then there is a number } c \text{ in } (a, b) \text{ such that } f'(c) = 0.\]
(b) Demonstrate that Rolle’s Theorem is true for the function \( f(x) = x^2 + 3x \) on the interval \([-3,0]\).

**Solution:** First verify that Rolle’s Theorem applies:

(i) \( f(x) = x^2 + 3x \) is continuous on \([-3,0]\) because it is a polynomial and is continuous everywhere.

(ii) \( f \) is differentiable on \((-3,0)\) because it is a polynomial and is differentiable everywhere.

(iii) \( f(-3) = (-3)^2 + 3(-3) = 0 \) and \( f(0) = 0^2 + 0 = 0 \) so \( f(-3) = f(0) \).

Now there should be a number \( c \) in \((-3,0)\) such that \( f'(c) = 0 \).

\[
\begin{align*}
f'(x) &= 2x + 3 \\
f'(c) &= 2c + 3
\end{align*}
\]

Solve

\[
\begin{align*}
f'(c) &= 0 \\
2c + 3 &= 0 \\
2c &= -3 \\
c &= -\frac{3}{2} = -1.5
\end{align*}
\]

and \(-3 < c = -1.5 < 0\).

(c) Use Rolle’s Theorem to show that \( f(x) = \sin(x) + 2x \) has exactly one root.

**Solution:** By inspection we see that \( f(0) = \sin(0) + 2(0) = 0 \), so there is at least one root. If there were another root at some \( x = b > 0 \), then since

(i) \( f \) is continuous on \([0,b]\),

(ii) \( f \) is differentiable on \((0,b)\),

(iii) \( f(0) = 0 \) and \( f(b) = 0 \) so \( f(0) = f(b) \),

Rolle’s theorem would tell us that there is a \( c \) in \((0,0)\) such that \( f'(c) = 0 \). But,

\[
\begin{align*}
f'(x) &= \cos(x) + 2 \\
f'(c) &= \cos(c) + 2
\end{align*}
\]

Try to solve \( f'(c) = 0 \):

\[
\begin{align*}
0 &= \cos(c) + 2 \\
-2 &= \cos(c)
\end{align*}
\]

There are no solutions because \( \cos(c) \) is always between \(-1\) and \(1\). So either Rolle’s Theorem is wrong or there is no such \( b \) where \( f(b) = 0 \). Rolle’s Theorem isn’t wrong, so there is no such \( b \).

Maybe there is a zero to the left, at some \( x = a < 0 \)? If that were true, then

(i) \( f \) is continuous on \([a,0]\),

(ii) \( f \) is differentiable on \((a,0)\),

(iii) \( f(0) = 0 \) and \( f(a) = 0 \) so \( f(0) = f(a) \),

so Rolle’s theorem would tell us that there is a \( c \) in \((a,0)\) such that \( f'(c) = 0 \). But,

\[
\begin{align*}
f'(x) &= \cos(x) + 2 \\
f'(c) &= \cos(c) + 2
\end{align*}
\]

Again, try to solve \( f'(c) = 0 \):

\[
\begin{align*}
0 &= \cos(c) + 2 \\
-2 &= \cos(c)
\end{align*}
\]

We reach the same contradiction; there are no solutions. So there is no point \( a < 0 \) such that \( f(a) = 0 \). This means that \( f \) has only root at \( x = 0 \).
9. (a) State the Mean Value Theorem.

**Solution:** If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$ then there is a number $c$ in $(a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

(b) Demonstrate that the Mean Value Theorem is true for the function $f(x) = 1/x$ on the interval $[0.5, 1]$.

**Solution:** First we verify that the Mean Value Theorem applies:

(i) $1/x$ is continuous on $[0.5, 1]$ since it is continuous everywhere except at 0 and 0 is not in the interval $[0.5, 1]$.

(ii) $1/x$ is differentiable on $(0.5, 1)$ because it is differentiable everywhere except at 0 and 0 is not in the interval $(0.5, 1)$.

Therefore, there should be a number $c$ in $(0.5, 1)$ such that $f(1) - f(0.5) = f'(c)(1 - 0.5)$.

\[
f'(x) = -\frac{1}{x^2}
\]

\[
f'(c) = -\frac{1}{c^2}
\]

\[
f(1) - f(0.5) = f'(c)(1 - 0.5)
\]

\[
\frac{1}{1} - \frac{1}{0.5} = -\frac{1}{c^2}(0.5)
\]

\[
-1 = -\frac{1}{2c^2}
\]

\[
2c^2 = 1
\]

\[
c^2 = \frac{1}{2}
\]

\[
c = \frac{1}{\sqrt{2}} \approx 0.707.
\]

10. Evaluate the following limits:

(a) $\lim_{x \to \infty} x^2 e^{-x^2}$

**Solution:** As $x \to \infty$, $x^2 \to \infty$ and $e^{-x^2} \to 0$, resulting in the indeterminate form $\infty \cdot 0$. Rewrite as a fraction and use L'Hospital's Rule:

\[
\lim_{x \to \infty} x^2 e^{-x^2} = \lim_{x \to \infty} \frac{x^2}{e^{x^2}}
\]

\[
(L) = \lim_{x \to \infty} \frac{2x}{(2x)e^{x^2}}
\]

\[
= \lim_{x \to \infty} \frac{1}{e^{x^2}}
\]

\[
= 0.
\]
(b) \( \lim_{x \to 0^+} \left( \frac{1}{x} \right)^x \)

**Solution:** As \( x \to 0^+ \), \( \frac{1}{x} \to \infty \), resulting in the indeterminate form \( \infty^0 \). Let

\[
y = \left( \frac{1}{x} \right)^x
\]

and consider

\[
\lim_{x \to 0^+} \ln(y) = \lim_{x \to 0^+} \ln \left( \frac{1}{x} \right)^x = \lim_{x \to 0^+} x \ln \left( \frac{1}{x} \right)
\]

Rewrite \( x \ln \left( \frac{1}{x} \right) \) as a fraction:

\[
\lim_{x \to 0^+} x \ln \left( \frac{1}{x} \right) = \lim_{x \to 0^+} \frac{\ln \left( \frac{1}{x} \right)}{\frac{1}{x}}
\]

As \( x \to 0^+ \), \( \frac{1}{x} \to \infty \) and \( \ln \left( \frac{1}{x} \right) \to \infty \), resulting in the indeterminate form \( \frac{\infty}{\infty} \). Use L'Hospital’s Rule:

\[
\lim_{x \to 0^+} \frac{\ln \left( \frac{1}{x} \right)}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{-\frac{1}{x^2}}{-\frac{1}{x}} = \lim_{x \to 0^+} \frac{1}{x} = \lim_{x \to 0^+} x = 0.
\]

Finally, if \( \ln(y) \to 0 \), then \( y = e^{\ln(y)} \to e^0 = 1 \). Therefore,

\[
\lim_{x \to 0^+} \left( \frac{1}{x} \right)^x = 1.
\]

11. Find \( f'(x), f''(x) \), vertical and horizontal asymptotes, critical points, intervals of increase and decrease, intervals of concavity, and sketch the graph (label the important points):

(a) \( f(x) = xe^{-x^2} \)

**Solution:**
There are no vertical asymptotes because the domain of \( f \) is \((-\infty, \infty)\). To find the horizontal asymptotes (if any), look at the limits as \( x \to \infty \) and \( x \to -\infty \) of \( f(x) \):

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} xe^{-x^2} = \lim_{x \to \infty} \frac{x}{e^{x^2}} \quad \left( \frac{\infty}{\infty} \text{ indeterminate form} \right)
\]

\[
(L) = \lim_{x \to \infty} \frac{1}{2xe^{x^2}} = 0.
\]

\[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} xe^{-x^2} = 0.
\]

There is one horizontal asymptote, \( y = 0 \).

To find the critical numbers of \( f \), first find the derivative of \( f \):

\[
f'(x) = (x)(-2x)e^{-x^2} + (1)e^{-x^2} = e^{-x^2} (1 - 2x^2).
\]

\( f' \) always exists, so set \( f' = 0 \).

\[
f'(x) = 0 \quad \Rightarrow \quad e^{-x^2} (1 - 2x^2) = 0
\]

\[
e^{-x^2} = 0 \quad \text{or} \quad 1 - 2x^2 = 0
\]

\[
e^{-x^2} \neq 0 \quad \Rightarrow \quad 1 = 2x^2 \quad \Rightarrow \quad x^2 = \frac{1}{2}
\]

\[
\pm \frac{1}{\sqrt{2}} = x
\]

Make a sign chart to find the intervals of increase and decrease of \( f \):

\[
f' \quad \oplus \quad \ominus \quad \oplus
\]

\[
\frac{-1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}
\]

From the chart, we see that \( f \) is decreasing on \((-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \infty)\) and increasing on \((-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\).

Determine the intervals of concavity by first finding the second derivative of \( f \):

\[
f''(x) = -2xe^{-x^2} (1 - 2x^2) + e^{-x^2} (-4x)
\]

\[
= e^{-x^2} (-2x + 4x^3 - 4x)
\]

\[
= e^{-x^2} (4x^3 - 6x)
\]

\( f'' \) always exists, so set \( f'' = 0 \). As above, since \( e^{-x^2} \neq 0 \) for any \( x \), we can just set \( 4x^3 - 6x = 0 \):

\[
4x^3 - 6x = 0
\]

\[
2x(2x^2 - 3) = 0
\]

\[
x = 0 \quad \text{or} \quad 2x^2 = 3
\]

\[
x^2 = \frac{3}{2}
\]

\[
x = \pm \sqrt{\frac{3}{2}}
\]
Make a sign chart to find the intervals of concavity of $f$:

\[
\begin{array}{c}
f' \\
\hline
\odot & \odot & \odot & \odot & \odot \\
\hline
-\sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}}
\end{array}
\]

From the chart, we see that $f$ is concave up on $\left(-\sqrt{\frac{3}{2}}, 0\right) \cup \left(\sqrt{\frac{3}{2}}, \infty\right)$ and concave down on $\left(-\infty, -\sqrt{\frac{3}{2}}\right) \cup \left(0, \sqrt{\frac{3}{2}}\right)$.

Graph of $f$:

(b) $f(x) = \frac{x - 1}{x^2 - 1}$

**Solution:**

Since the denominator $x^2 - 1 = 0$ when $x = -1$ or $x = 1$, these are candidates for vertical asymptotes. Check the limits:

\[
\lim_{x \to -1^+} \frac{x - 1}{x^2 - 1} = \lim_{x \to -1^+} \frac{x - 1}{(x - 1)(x + 1)} = \lim_{x \to -1^+} \frac{1}{x + 1} = \infty.
\]

\[
\lim_{x \to -1} \frac{x - 1}{x^2 - 1} = \lim_{x \to -1} \frac{x - 1}{(x - 1)(x + 1)} = \lim_{x \to -1} \frac{1}{x + 1} = \frac{1}{2}.
\]

Thus there is only one vertical asymptote, $x = -1$.

Check for horizontal asymptotes:

\[
\lim_{x \to \infty} \frac{x - 1}{x^2 - 1} = \lim_{x \to \infty} \frac{x - 1}{x^2} = \lim_{x \to \infty} \frac{1 - \frac{1}{x^2}}{1 - \frac{1}{x^2}} = 0.
\]
\[
\lim_{x \to -\infty} \frac{x - 1}{x^2 - 1} = \lim_{x \to -\infty} \frac{x}{x^2} - \frac{1}{x^2} = \lim_{x \to -\infty} \frac{1 - \frac{1}{x^2}}{1 - \frac{1}{x^2}} = 0 - 0 = 0.
\]

There is one horizontal asymptote, \( y = 0 \).

Find the critical numbers of \( f \):

\[
f'(x) = \frac{(x^2 - 1)(1) - (x - 1)(2x)}{(x^2 - 1)^2} = \frac{x^2 - 1 - 2x^2 + 2x}{(x^2 - 1)^2} = \frac{-x^2 - 2x + 1}{(x^2 - 1)^2} = \frac{(x-1)^2}{(x^2 - 1)^2}
\]

Since \( x \neq 1 \), this simplifies to

\[
f'(x) = -\frac{(x-1)^2}{(x^2 - 1)^2} = -\frac{(x-1)(x-1)}{(x-1)(x+1))^2} = \frac{-1}{(x+1)^2}
\]

The derivative does not exist when \( x + 1 = 0 \), which is true for \( x = -1 \). However, \(-1\) is not in the domain of \( f \). The derivative is never equal to 0 since \( x = 1 \) is also not in the domain of \( f \).

Make a sign chart to find the intervals of increase and decrease of \( f \):

\[
f' \quad \odot \quad \odot \quad \odot
\]

\( f \) is decreasing on its entire domain, \((-\infty, -1) \cup (-1, 1) \cup (1, \infty)\).

Determine the intervals of concavity by looking at \( f'' \):

\[
f''(x) = \frac{0 - (-1)(2)(x + 1)}{(x + 1)^4} = \frac{2}{(x + 1)^3}
\]

\( f'' \) does not exist when \( x = -1 \), but \( x = -1 \) is not in the domain of \( f \). \( f'' \) is never 0.

Make a sign chart for \( f'' \):

\[
f'' \quad \odot \quad \oplus \quad \oplus
\]

\( f \) is concave up on \((-1, 1) \cup (1, \infty)\) and concave down on \((-\infty, -1)\).
12. For each part, if the statement is always true, mark it T. If the statement is sometimes false, mark it F. For each question, write a careful and clear justification or describe a counterexample.

(a) If $f$ and $g$ are increasing on the interval $(a,b)$, then $fg$ is also increasing on $(a,b)$.

Solution: False. Let $f(x) = x$ and $g(x) = x$. $f$ and $g$ are always increasing. In particular, they are increasing on the interval $(-1,0)$. But $fg = (x)(x) = x^2$ is a parabola and is decreasing on $(-1,0)$.

(b) $\frac{d}{dx} (\ln(10)) = \frac{1}{10}$.

Solution: False. $\ln(10)$ is a constant, so $\frac{d}{dx} (\ln(10)) = 0$.

(c) A continuous function defined on all points of a closed interval always has a global maximum and a global minimum.

Solution: True. This is the statement of the Extreme Value Theorem.

(d) If $f' > 0$ on an interval, the function is concave up on the interval.

Solution: False. Let $f(x) = -x^2$. $f'(x) = -2x$ and $f' > 0$ for $x < 0$. However, $f''(x) = -2 < 0$, so $f$ is always concave down.