Section 2.3: Inverse of a Square Matrix

Definition: The inverse of an \( n \times n \) matrix \( A \) is a matrix \( B \) such that

\[
AB = BA = I_n.
\]

If \( B \) exists, we write \( B = A^{-1} \). If a matrix \( A \) does not have an inverse, we call \( A \) singular. If the inverse of \( A \) does exist, we call \( A \) nonsingular.

Example 1: Determine if the following matrices are inverses of each other.

\[
A = \begin{bmatrix}
-2 & 4 \\
-3 & 7 \\
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
-7/2 & 2 \\
-3/2 & 1 \\
\end{bmatrix}
\]
Finding Inverses

Let $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

**Guass-Jordan Elimination:** Use Guassian elimination to get 1’s on the diagonal 0’s below and above the 1’s.

**To find the inverse $A^{-1}$ of a matrix $A$:**
1. Form the augmented matrix $[A|I]$.
2. Use elementary row operations (Guass-Jordan method) to reduce $[A|I]$ to $[I|B]$, if possible.
3. If step 2 is possible, then $B = A^{-1}$. If not, then $A$ is singular.
Elementary Row Operations:

1. **Interchange** the \( i \)th row and the \( j \)th row.
   
   \( R_i \leftrightarrow R_j \)

2. **Multiply** each member of the \( i \)th row by a non-zero constant \( k \).
   
   \( kR_i \rightarrow R_i \)

3. **Replace** each element in the \( i \)th row with the corresponding element of the \( i \)th row plus \( k \) times the \( j \)th row.
   
   \( R_i + kR_j \rightarrow R_i \)

**Example 2:** Find the inverse, if it exists, of \( A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \).
Example 3: Find the inverse, if it exists, of \( B = \begin{bmatrix} 4 & -2 & 3 \\ 8 & -3 & 5 \\ 7 & -2 & 4 \end{bmatrix} \).
Example 4: Find the inverse, if it exists, of $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 4 \\ 1 & 5 & 10 \end{bmatrix}$. 
Example 5: Application to systems of linear equations. Use an inverse of a matrix to solve the following system of equations.

\[
\begin{align*}
4x - 2y + 3z &= 3 \\
8x - 3y + 5z &= 2 \\
7x - 2y + 4z &= 0 
\end{align*}
\]