Section F.3: Annuities and Sinking Funds

Definitions: An annuity is a sequence of equal payments made at regular intervals of time. An ordinary annuity is one in which the payments are made at the end of the compounding period. The term of an annuity if the time from the beginning of the first period to the end of the last period.

Examples: regular deposits in savings or retirement accounts, monthly mortgage payments, sinking funds, which is just an account, often used by corporations, that is established to accumulate funds for some future need.

\[ F = \text{the future value of an ordinary annuity.} \]
\[ R = \text{payment made at the end of each compounding period.} \]
\[ i = \frac{r}{m} \text{ is the interest rate per period.} \]
\[ n = mt \text{ is the number of payments.} \]

**Future Value of an Ordinary Annuity**

\[ F = R \times \frac{[(1 + i)^n - 1]}{i} \]
Example 1: Finding Future Value A woman pays $130 at each quarter's end into an account earning 5.4% annual interest compounded quarterly.

(a) How much money will be in the account after 10 years?
(b) How much of this is from deposits? from interest?

\[ R = 130 \quad \text{quarterly} \quad m = 4 \quad t = 10 \quad r = 0.054 \]
\[ m = 4 \quad \therefore i = \frac{0.054}{4} \quad n = 4 \times 10 = 40 \]
\[ F = 130 \left[ \left(1 + \frac{0.054}{4} \right)^{40} - 1 \right] \quad (0.054) \]
\[ = 6835.29 \]

From deposits = 130 \times 40 = \$5200
From interest = 6835.29 - 5200 = \$1635.29.
Example 2: Finding Payment Amount You create a sinking fund that you wish to be worth $20,000 to use as a down payment on a house at the end of 5 years. The account earns 6.4% interest compounded semiannually.

(a) How much should you deposit at the end of every 6 months in order to have $20,000 in 5 years?

(b) How much interest was earned in the second half of the fourth year?

\[
F = 0.064 \\
N = 2 \quad \text{(semi annually twice a year)} \\
T = 5 \\
N = 5 \times 2 = 10
\]

\[
F = R \left[ \left(1 + \frac{i}{2}\right)^{2t} - 1 \right] = \frac{20,000}{(1 + \left(\frac{0.064}{2}\right)^{10})} \cdot R
\]

\[
R = 1728.60 \quad \text{\$}.
\]

\[
F \text{ at the end of } 3\text{rd.5 year} \\
N = 3 \times 2 + 1 = 7
\]

\[
F = 1728.60 \left[ \left(1 + \frac{0.064}{2}\right)^{7} - 1 \right] \div (0.064) \quad \frac{(0.064)^{12}}{2}
\]

\[
F = 13325.82 \quad \text{\$}
\]

\[
F_{4} - F_{3.5} = 2154.99 \quad \text{\$} \quad \text{(This includes a payment made in 2nd half of the year)}
\]

\[
\Rightarrow \text{Intearned} = 2154.99 - 1728.60 = 426.398 \quad \text{\$}.
\]
Future Value of Annuity.

\[ F = R \left[ \frac{(1+i)^n - 1}{i} \right] \]

- \( F \) = Future value
- \( R \) = payment made at the end of each compounding period 'Just like Rent'
- \( i \) = \( r/m \) interest rate/period
- \( n \) = \( m \times t \) is number of payments.

The idea of this annuity is to save for the future.

Examples
- College education
- Down payment of a car/house
- Retirement
- etc

\text{x} Key thing to remember is that one is saving some money every period. This is different from compound interest where the money is put in once as a lumpsum amount and it gathers interest.
3. College Education

$100 Every month

\[ R = 100, \ m = 12, \ r = 6\% = 0.06 \]

\[ F = R \left[ \frac{(1+i)^n - 1}{i} \right] \]

\[ t=0 \]

Six years old

\[ t=12 \]

18 years old.

\[ n = m \cdot t = 12 \times 12 = 144 \]

\[ i = \frac{r}{m} = \frac{0.06}{12} = 0.005 \]

\[ F = 100 \left[ \frac{(1 + 0.06)^{144} - 1}{(0.06)^{12}} \right] = 21015.01 \$

4. Financing for car down payment

\[ F = 5000, \ m = 12, \ r = 0.04 \ t = 4 \text{ years} \]

Find R

\[ i = \frac{r}{m} = \frac{0.04}{12}, \ n = m \cdot t = 12 \times 4 = 48 \]

\[ F = R \left[ \frac{(1+i)^n - 1}{i} \right] = R \left[ \frac{(1+0.04)^{48} - 1}{(0.04)^{12}} \right] = 5000 \]

\[ R = 9623 \$ \text{/month} \] have to be saved.
5. Retirement Account:

2000 $ every year.

\[ M = 1, \quad r = 0.09, \quad n = m \cdot t \]

\[ R = 2000, \quad t = 40 = 1 \cdot 40 = 40. \]

\[ F = R \cdot \frac{(1+i)^n-1}{i} \]

\[ = 2000 \cdot \frac{(1+0.09)^{40}-1}{0.09} = 675764.89 $. \]

How much interest is earned.

So each year for 40 years the individual put 2000 $.

\[ \therefore \text{Total money he put over 40 years} \]

\[ = 2000 \times 40 = 80,000. \]

At the end of 40 years he has accumulated 675,764.89 $.

\[ \therefore \text{Total interest earned} = 675,764.89 - 80,000 \]

\[ = 595,764.89 \]
6. College Education

A) \( F = 200,000 \)
\( t = 18 \) years
\( m = 12 \)
\( n = m \cdot t = 18 \times 12 = 216 \)
\( i = \frac{0.09}{12} \)
\( R = ? \)

\[
F = R \left[ \frac{(1+i)^n - 1}{i} \right]
\]

\[
200,000 = R \left[ \frac{(1 + \frac{0.09}{12})^{216}}{(0.09)} \right] - 1
\]

\[
\therefore R = 372.89
\]

B) If the parents make a payment of $372.89/month for 18 years, total money they would have put in is

\[
372.89 \times (18 \times 12) = 80544.17
\]

So total interest they earned in 18 years

\[
= 200,000 - 80544.17 = 119,455.83\text{.}$

C) Equity in 10 years.

So \( F = ? \) after 10 years \( n = m \cdot t = 12 \times 10 = 120 \)
\( i = \frac{r}{m} = \frac{0.09}{12} \)
\( R = 372.89 \)

\[
\therefore F = \left( \frac{1 + \frac{0.09}{12}}{12} \right)^{120} - 1
\]

\[
372.89 = 72159.54\text{.}$
D. You want to know how much interest was earned in the 10th year.

We already know how much money is in the account at the end of 10 years: 72,159.54 $.

So now find out what is the money after 9 years.

\[
\begin{align*}
\text{\( n = 9 \times 12 \)} & \quad \text{\( i = \frac{0.09}{12} \)} \\
\text{\( R = 372.89 \)} & \quad \text{\( F = ? \)}
\end{align*}
\]

\[
F = \frac{(1+i)^n - 1}{i} \quad R = \frac{(1+\frac{0.09}{12})^{108}}{(0.09)} - 1 \quad 372.89
\]

\[
= 61,707.04 $.
\]

So at the end of 9 years the account has 61,707.04 $. This is the amount of money at the beginning of 10th year.

So total money in 10th = 72,159.54 - 61,707.04

= 10,452.5 $.

This portion has the monthly payments the parents made during the 10th year = 12 x 372.89 = 4,474.68 $.

So total interest earned in 10th year

= 10,452.5 - 4,474.68 = 5,977.82 $.