Assignment 1: Module Theory

You can work on this assignment in groups. If you do so, please indicate on this page the names of people you have worked with. Everybody needs to hand in his/her own copy in class on February 23.

Name: __________________________

Other group members: ____________________________________________

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1. Show that $C \otimes_R C$ and $C \otimes_C C$ are both left $R$-modules but are not isomorphic as $R$-modules.

2. Let $A$ be a finite $\mathbb{Z}$-module of order $n$ and let $p^k$ be the largest power of the prime $p$ dividing $n$. Let $B$ be the $p$-Sylow subgroup of $A$ (why is there only one $p$-Sylow subgroup?). Prove that $\mathbb{Z}/p^k \mathbb{Z} \otimes_\mathbb{Z} A$ is isomorphic to $B$.

3. Let $R$ be an integral domain and let $I$ be a principal ideal in $R$. Prove that the $R$-module $I \otimes_R I$ has no nonzero torsion elements. (Recall that $m \in I \otimes_R I$ is a torsion element, if there exists $r \in R$ such that $r \neq 0$ and $rm = 0$.)

4. Show that $\mathbb{Z}[i] \otimes_\mathbb{Z} \mathbb{R}$ and $\mathbb{C}$ are isomorphic as rings, where the multiplication in the ring $\mathbb{Z}[i] \otimes_\mathbb{Z} \mathbb{R}$ is given by Proposition 21 of Chapter 10.

5. Let $R$ be a ring with identity, let $M,N$ be left $R$-modules and let $D$ be a right $R$-module. Let $\varphi : M \to N$ be a surjective $R$-module homomorphism and denote by $1 : D \to D$ the identity homomorphism.

(a) Show that the induced group homomorphism $1 \otimes \varphi : D \otimes_R M \to D \otimes_R N$ is surjective.

(b) Show that for an injective homomorphism $\psi : M \to N$ the induced homomorphism $1 \otimes \psi$ is not always injective. [Hint: Give a counterexample using $\psi : \mathbb{Z} \to \mathbb{Q}$.]

6. Let $R$ be a ring with identity and let $e \in R$ be an idempotent (that is $e^2 = e$). From assignment #3 from the Fall we know that $(1 - e)$ is also an idempotent and that $e$ and $(1 - e)$ are orthogonal. Show that the module $Re \oplus R(1 - e)$ is free, thus, in particular, $Re$ is projective.

7. Let $R$ be a ring with identity, $A,B,M$ left $R$-modules, and let $f : A \to M, g : B \to M$ be homomorphisms. Define the fiber product of $f$ and $g$ as

$$X = \{(a,b) \mid a \in A, b \in B, \text{ such that } f(a) = g(b)\},$$

and define the projections $\pi_1(a,b) = a$ and $\pi_2(a,b) = b$.

(a) Show that $X$ is a submodule of $A \oplus B$ and that there is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_2} & B \\
\downarrow{\pi_1} & & \downarrow{g} \\
A & \xrightarrow{f} & M \\
\end{array}
\]

(b) Show that if $f$ is injective then $\pi_2$ is injective.

(c) Show that if $f$ is surjective then $\pi_2$ is surjective.

(d) Now suppose $0 \to N \xrightarrow{h} A \xrightarrow{f} M \to 0$ is a short exact sequence and define $h' : N \to X$ by $h'(n) = (h(n),0)$. Show that the following diagram is commutative with exact rows.

\[
\begin{array}{ccc}
0 & \xrightarrow{1_N} & N & \xrightarrow{h'} & X & \xrightarrow{\pi_2} & B & \to & 0 \\
& & \downarrow{\pi_1} & & \downarrow{g} & & \downarrow{f} & & \downarrow{} & & \downarrow{} \\
0 & \xrightarrow{} & N & \xrightarrow{h} & A & \xrightarrow{f} & M & \to & 0
\end{array}
\]

Note: This exercise shows that, for any homomorphism $g : B \to M$, the fiber product defines a map from the extensions of $M$ by $N$ (lower row of the diagram) to the extensions of $B$ by $N$ (upper row of the diagram).