35. MATLAB Student Version 4.0 uses 57,771 flops for inv A and 14,269,045 flops for the inverse formula. The \texttt{inv(A)} command requires only about 0.4% of the operations for the inverse formula.

Chapter 3  SUPPLEMENTARY EXERCISES

1. a. True. The columns of A are linearly dependent.
   b. True. See Exercise 30 in Section 3.2.
   c. False. See Theorem 3(c); in this case \( \det 5A = 5^3 \det A \).

   d. False. Consider \( A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \text{ and } A + B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \). 

   e. False. By Theorem 6, \( \det A^3 = 2^3 \).
   f. False. See Theorem 3(b).
   g. True. See Theorem 3(c).
   h. True. See Theorem 3(a).
   i. False. See Theorem 5.
   j. False. See Theorem 3(c); this statement is false for \( n \times n \) invertible matrices with \( n \) an even integer.
   k. True. See Theorems 6 and 5; \( \det A^T A = (\det A)^2 \).
   l. False. The coefficient matrix must be invertible.
   m. False. The area of the \textbf{triangle} is 5.
   n. True. See Theorem 6; \( \det A^3 = (\det A)^3 \).
   o. False. See Exercise 31 in Section 3.2.

\[
\begin{vmatrix}
12 & 13 & 14 \\
15 & 16 & 17 \\
18 & 19 & 20
\end{vmatrix} = 
\begin{vmatrix}
12 & 13 & 14 \\
3 & 3 & 3 \\
6 & 6 & 6
\end{vmatrix} = 0
\]

3. \[ \begin{vmatrix}
1 & a & b+c \\
1 & a & b+c \\
1 & a & b+c
\end{vmatrix} = 
\begin{vmatrix}
1 & a & b+c \\
0 & b-a & a-b \\
0 & c-a & a-c
\end{vmatrix} = (b-a)(c-a) \begin{vmatrix}
0 & 1 & -1 \\
1 & 1 & -1
\end{vmatrix} = 0
\]

4. \[
\begin{vmatrix}
a & b & c \\
a+x & b+x & c+x \\
a+y & b+y & c+y
\end{vmatrix} = 
\begin{vmatrix}
a & b & c \\
x & x & x = xy \\
y & y & y
\end{vmatrix} = 0
\]

\[
\begin{vmatrix}
9 & 1 & 9 & 9 \\
9 & 0 & 9 & 2 \\
4 & 0 & 5 & 0 \\
9 & 0 & 3 & 9 \\
6 & 0 & 7 & 0
\end{vmatrix} = (-1)
\begin{vmatrix}
9 & 9 & 9 & 2 \\
4 & 0 & 5 & 0 \\
9 & 3 & 9 & 0 \\
6 & 0 & 7 & 0
\end{vmatrix} = (-1)(-2)
\begin{vmatrix}
4 & 5 \\
6 & 7
\end{vmatrix} = (-1)(-2)(3)(-2) = -12
\]
15. This problem is equivalent to finding a basis for \( \text{Col } A \), where \( A = [v_1 \ v_2 \ v_3 \ v_4 \ v_5] \). Since the reduced echelon form of \( A \) is

\[
\begin{bmatrix}
1 & 0 & -3 & 1 & 2 \\
0 & 1 & -4 & -3 & 1 \\
-3 & 2 & 1 & -8 & -6 \\
2 & -3 & 6 & 7 & 9
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & -3 & 0 & 4 \\
0 & 1 & -4 & 0 & -5 \\
0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

we see that the first, second, and fourth columns of \( A \) are its pivot columns. Thus a basis for the space spanned by the given vectors is

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \\ 6 \end{bmatrix} \right\}
\]

16. This problem is equivalent to finding a basis for \( \text{Col } A \), where \( A = [v_1 \ v_2 \ v_3 \ v_4 \ v_5] \). Since the reduced echelon form of \( A \) is

\[
\begin{bmatrix}
1 & -2 & 6 & 5 & 0 \\
0 & 1 & -1 & -3 & 3 \\
0 & -1 & 2 & 3 & -1 \\
1 & 1 & -1 & -4 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & -1 & -2 \\
0 & 1 & 0 & -3 & 5 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

we see that the first, second, and third columns of \( A \) are its pivot columns. Thus a basis for the space spanned by the given vectors is

\[
\left\{ \begin{bmatrix} 1 \\ -2 \\ 6 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}
\]

4.5 p. 261

13. The matrix \( A \) is in echelon form. There are three pivot columns, so the dimension of \( \text{Col } A \) is 3. There are two columns without pivots, so the equation \( Ax = 0 \) has two free variables. Thus the dimension of \( \text{Nul } A \) is 2.

14. The matrix \( A \) is in echelon form. There are three pivot columns, so the dimension of \( \text{Col } A \) is 3. There are three columns without pivots, so the equation \( Ax = 0 \) has three free variables. Thus the dimension of \( \text{Nul } A \) is 3.

15. The matrix \( A \) is in echelon form. There are two pivot columns, so the dimension of \( \text{Col } A \) is 2. There are two columns without pivots, so the equation \( Ax = 0 \) has two free variables. Thus the dimension of \( \text{Nul } A \) is 2.
5.1 p. 308

9. For \( \lambda = 1 \):

\[
A - I = \begin{bmatrix}
5 & 0 \\
2 & 1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
4 & 0 \\
2 & 0
\end{bmatrix}
\]

The augmented matrix for \((A - I)x = 0\) is

\[
\begin{bmatrix}
4 & 0 & 0 \\
2 & 0 & 0
\end{bmatrix}
\]

Thus \(x_1 = 0\) and \(x_2\) is free. The general solution of \((A - I)x = 0\) is \(x_2e_2\), where \(e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\), and so \(e_2\) is a basis for the eigenspace corresponding to the eigenvalue 1.

For \( \lambda = 5 \):

\[
A - 5I = \begin{bmatrix}
5 & 0 \\
2 & 1 \\
0 & 5
\end{bmatrix} \begin{bmatrix}
5 & 0 \\
0 & 5
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
2 & -4
\end{bmatrix}
\]

The equation \((A - 5I)x = 0\) leads to \(2x_1 - 4x_2 = 0\), so that \(x_1 = 2x_2\) and \(x_2\) is free. The general solution is

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
\]

So \(x_2\) is a basis for the eigenspace.

10. For \( \lambda = 4 \):

\[
A - 4I = \begin{bmatrix}
10 & -9 \\
4 & -2
\end{bmatrix} \begin{bmatrix}
4 & 0 \\
0 & 4
\end{bmatrix} = \begin{bmatrix}
6 & -9 \\
4 & -6
\end{bmatrix}
\]

The augmented matrix for \((A - 4I)x = 0\) is

\[
\begin{bmatrix}
6 & -9 & 0 \\
4 & -6 & 0
\end{bmatrix} \begin{bmatrix}
1 & -9/6 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Thus \(x_1 = (3/2)x_2\) and \(x_2\) is free. The general solution is

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (3/2)x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} (3/2) \\ 1 \end{bmatrix}.
\]

A basis for the eigenspace corresponding to 4 is \(x_2\). Another choice is \(x_2\).

5.2 p. 317

10. \(\det(A - \lambda I) = \det \begin{bmatrix}
0 - \lambda & 3 & 1 \\
3 & 0 - \lambda & 2 \\
1 & 2 & 0 - \lambda
\end{bmatrix}\)

From the special formula for 3x3 determinants, the characteristic polynomial is

\[
\det(A - \lambda I) = (-\lambda)(-\lambda)(-\lambda) + 3 \cdot 2 \cdot 1 + 1 \cdot 3 \cdot 2 - 1 \cdot (-\lambda) \cdot 1 - 2 \cdot 2 \cdot (-\lambda) - (-\lambda) \cdot 3 \cdot 2
\]

\[
= -\lambda^3 + 6 + 6 + 6 + 4\lambda + 8\lambda = -\lambda^3 + 14\lambda + 12
\]
1. \[ A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}, \quad A - \lambda I = \begin{bmatrix} 2 - \lambda & 7 \\ 7 & 2 - \lambda \end{bmatrix} \] The characteristic polynomial is
\[
\det(A - \lambda I) = (2 - \lambda)^2 - 7^2 = 4 - 4\lambda + \lambda^2 - 49 = \lambda^2 - 4\lambda - 45
\]
In factored form, the characteristic equation is \((\lambda - 9)(\lambda + 5) = 0\), so the eigenvalues of \(A\) are 9 and -5.

2. \[ A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}, \quad A - \lambda I = \begin{bmatrix} 5 - \lambda & 3 \\ 3 & 5 - \lambda \end{bmatrix} \] The characteristic polynomial is
\[
\det(A - \lambda I) = (5 - \lambda)(5 - \lambda) - 3 \cdot 3 = \lambda^2 - 10\lambda + 16
\]
Since \(\lambda^2 - 10\lambda + 16 = (\lambda - 8)(\lambda - 2)\), the eigenvalues of \(A\) are 8 and 2.

3. \[ A = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}, \quad A - \lambda I = \begin{bmatrix} 3 - \lambda & -2 \\ 1 & -1 - \lambda \end{bmatrix} \] The characteristic polynomial is
\[
\det(A - \lambda I) = (3 - \lambda)(-1 - \lambda) - (-2)(1) = \lambda^2 - 2\lambda - 1
\]
Use the quadratic formula to solve the characteristic equation and find the eigenvalues:
\[
\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 + 4}}{2} = 1 \pm \sqrt{2}
\]

4. \[ A = \begin{bmatrix} 5 & -3 \\ -4 & 3 \end{bmatrix}, \quad A - \lambda I = \begin{bmatrix} 5 - \lambda & -3 \\ -4 & 3 - \lambda \end{bmatrix} \] The characteristic polynomial of \(A\) is
\[
\det(A - \lambda I) = (5 - \lambda)(3 - \lambda) - (-3)(-4) = \lambda^2 - 8\lambda + 3
\]
Use the quadratic formula to solve the characteristic equation and find the eigenvalues:
\[
\lambda = \frac{8 \pm \sqrt{64 - 4(3)}}{2} = \frac{8 \pm 2\sqrt{13}}{2} = 4 \pm \sqrt{13}
\]

9. \[ \det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & 0 & -1 \\ 2 & 3 - \lambda & -1 \\ 0 & 6 & 0 - \lambda \end{bmatrix} \] From the special formula for 3x3 determinants, the characteristic polynomial is
\[
\det(A - \lambda I) = (1 - \lambda)(3 - \lambda)(-\lambda) + 0 + (-1)(2)(6) - 0 - (6)(1 - \lambda) = 0
\]
\[
= (\lambda^2 - 4\lambda + 3)(-\lambda) - 12 + 6(1 - \lambda)
\]
\[
= -\lambda^3 + 4\lambda^2 - 3\lambda - 12 + 6 - 6\lambda
\]
\[
= -\lambda^3 + 4\lambda^2 - 9\lambda - 6
\]
(This polynomial has one irrational zero and two imaginary zeros.) Another way to evaluate the determinant is to interchange rows 1 and 2 (which reverses the sign of the determinant) and then make one row replacement:
\[
\det \begin{bmatrix} 1 - \lambda & 0 & -1 \\ 2 & 3 - \lambda & -1 \\ 0 & 6 & 0 - \lambda \end{bmatrix} = -\det \begin{bmatrix} 2 & 3 - \lambda & -1 \\ 0 & 6 & 0 - \lambda \\ 1 - \lambda & 0 & -1 \end{bmatrix}
\]
\[
= -\det \begin{bmatrix} 2 & 3 - \lambda & -1 \\ 0 & 6 & 0 - \lambda \\ 0 & 0 + (.5\lambda - .5)(3 - \lambda) & -1 + (.5\lambda - .5)(-1) \end{bmatrix}
\]
Next, expand by cofactors down the first column. The quantity above equals
\[
-2\det \begin{bmatrix} (.5\lambda - .5)(3 - \lambda) & -5 & 5 - .5\lambda \\ 6 & 6 & 6 \end{bmatrix} = -2[(.5\lambda - .5)(3 - \lambda)(-\lambda) - (3 - \lambda)(.5\lambda - .5)(6)]
\]
\[
= (1 - \lambda)(3 - \lambda)(-\lambda) - (1 + \lambda)(6) = (\lambda^2 - 4\lambda + 3)(-\lambda) - 6 - 6\lambda = -\lambda^3 + 4\lambda^2 - 9\lambda - 6
\]

# 10 is on page 1
1. The matrix $B$ is in echelon form. There are two pivot columns, so the dimension of $\text{Col } A$ is 2. There are two pivot rows, so the dimension of $\text{Row } A$ is 2. There are two columns without pivots, so the equation $Ax = 0$ has two free variables. Thus the dimension of $\text{Nul } A$ is 2. A basis for $\text{Col } A$ is the pivot columns of $A$:

$$
\begin{bmatrix}
1 & -4 \\
-1 & 2 \\
5 & -6
\end{bmatrix}.
$$

A basis for $\text{Row } A$ is the pivot rows of $B$: $\{(1,0,-1,5),(0,-2,5,-6)\}$. To find a basis for $\text{Nul } A$ row reduce to reduced echelon form:

$$
A \sim \begin{bmatrix}
1 & 0 & -1 & 5 \\
0 & 1 & -5/2 & 3
\end{bmatrix}.
$$

The solution to $Ax = 0$ in terms of free variables is $x_1 = x_3 - 5x_4$, $x_2 = (5/2)x_3 - 3x_4$ with $x_3$ and $x_4$ free. Thus a basis for $\text{Nul } A$ is

$$
\begin{bmatrix}
1 & -5 \\
5/2 & -3 \\
1 & 0 \\
0 & 1
\end{bmatrix}.
$$

2. The matrix $B$ is in echelon form. There are three pivot columns, so the dimension of $\text{Col } A$ is 3. There are three pivot rows, so the dimension of $\text{Row } A$ is 3. There are two columns without pivots, so the equation $Ax = 0$ has two free variables. Thus the dimension of $\text{Nul } A$ is 2. A basis for $\text{Col } A$ is the pivot columns of $A$:

$$
\begin{bmatrix}
1 & 4 & 9 \\
-2 & -6 & -10 \\
-3 & -6 & -3 \\
3 & 4 & 0
\end{bmatrix}.
$$

A basis for $\text{Row } A$ is the pivot rows of $B$: $\{(1,-3,0,5,-7),(0,0,2,-3,8),(0,0,0,0,5)\}$. To find a basis for $\text{Nul } A$ row reduce to reduced echelon form:

$$
A \sim \begin{bmatrix}
1 & -3 & 0 & 5 & 0 \\
0 & 0 & 1 & -3/2 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

The solution to $Ax = 0$ in terms of free variables is $x_1 = 3x_2 - 5x_4$, $x_3 = (3/2)x_4$, $x_5 = 0$, with $x_2$ and $x_4$ free. Thus a basis for $\text{Nul } A$ is

$$
\begin{bmatrix}
3 & -5 \\
1 & 0 \\
0 & 3/2 \\
0 & 1 \\
0 & 0
\end{bmatrix}.$$
3. The matrix $B$ is in echelon form. There are three pivot columns, so the dimension of $\text{Col} \ A$ is 3. There are three pivot rows, so the dimension of $\text{Row} \ A$ is 3. There are two columns without pivots, so the equation $Ax = 0$ has two free variables. Thus the dimension of $\text{Nul} \ A$ is 2. A basis for $\text{Col} \ A$ is the pivot columns of $A$:

$$\begin{bmatrix} 2 & 6 & 2 \\ -2 & -3 & -3 \\ 4 & 9 & 5 \\ -2 & 3 & -4 \end{bmatrix}$$

A basis for $\text{Row} \ A$ is the pivot rows of $B$: $\{(2,-3,6,2,5),(0,0,3,-1,1),(0,0,0,1,3)\}$. To find a basis for $\text{Nul} \ A$ row reduce to reduced echelon form:

$$A \sim \begin{bmatrix} 1 & -3/2 & 0 & 0 & -9/2 \\ 0 & 0 & 1 & 0 & 4/3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution to $Ax = 0$ in terms of free variables is $x_1 = (3/2)x_2 + (9/2)x_5$, $x_3 = -(4/3)x_5$, $x_4 = -3x_5$, with $x_2$ and $x_5$ free. Thus a basis for $\text{Nul} \ A$ is

$$\begin{bmatrix} 3/2 & 9/2 \\ 1 & 0 \\ 0 & -4/3 \\ 0 & -3 \\ 0 & 1 \end{bmatrix}.$$

4. The matrix $B$ is in echelon form. There are three pivot columns, so the dimension of $\text{Col} \ A$ is 3. There are three pivot rows, so the dimension of $\text{Row} \ A$ is 3. There are three columns without pivots, so the equation $Ax = 0$ has three free variables. Thus the dimension of $\text{Nul} \ A$ is 3. A basis for $\text{Col} \ A$ is the pivot columns of $A$:

$$\begin{bmatrix} 1 & 1 & 7 \\ 1 & 2 & 10 \\ 1 & -1 & 1 \\ 1 & -3 & -5 \\ 1 & -2 & 0 \end{bmatrix}.$$

A basis for $\text{Row} \ A$ is the pivot rows of $B$:

$$\{(1,1,-3,7,9,-9),(0,1,-1,3,4,-3),(0,0,0,1,-1,-2)\}.$$ To find a basis for $\text{Nul} \ A$ row reduce to reduced echelon form:

$$A \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 9 & 2 \\ 0 & 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
The solution to $Ax = 0$ in terms of free variables is $x_1 = 2x_3 - 9x_5 - 2x_6$, $x_2 = x_3 - 7x_5 - 3x_6$, $x_4 = x_5 + 2x_6$, with $x_3$, $x_5$, and $x_6$ free. Thus a basis for $\text{Nul } A$ is

$$\begin{bmatrix} 2 & -9 & -2 \\ 1 & -7 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

5. By the Rank Theorem, $\dim \text{Nul } A = 8 - \text{rank } A = 8 - 3 = 5$. Since $\dim \text{Row } A = \text{rank } A$, $\text{dim Row } A = 3$.
   Since $\text{rank } A^T = \text{dim Col } A^T = \dim \text{Row } A$, $\text{rank } A^T = 3$.

6. By the Rank Theorem, $\dim \text{Nul } A = 3 - \text{rank } A = 3 - 3 = 0$. Since $\dim \text{Row } A = \text{rank } A$, $\dim \text{Row } A = 3$.
   Since $\text{rank } A^T = \text{dim Col } A^T = \dim \text{Row } A$, $\text{rank } A^T = 3$.

7. Yes, $\text{Col } A = \mathbb{R}^4$. Since $A$ has four pivot columns, $\dim \text{Col } A = 4$. Thus $\text{Col } A$ is a four-dimensional subspace of $\mathbb{R}^4$, and $\text{Col } A = \mathbb{R}^4$.
   No, $\text{Nul } A \neq \mathbb{R}^3$. It is true that $\dim \text{Nul } A = 3$, but $\text{Nul } A$ is a subspace of $\mathbb{R}^7$.

8. Since $A$ has four pivot columns, $\text{rank } A = 4$, and $\dim \text{Nul } A = 6 - \text{rank } A = 6 - 4 = 2$.
   No, $\text{Col } A \neq \mathbb{R}^4$. It is true that $\dim \text{Col } A = \text{rank } A = 4$, but $\text{Col } A$ is a subspace of $\mathbb{R}^5$.

9. Since $\dim \text{Nul } A = 4$, $\text{rank } A = 6 - \dim \text{Nul } A = 6 - 4 = 2$. So $\dim \text{Col } A = \text{rank } A = 2$.

10. Since $\dim \text{Nul } A = 5$, $\text{rank } A = 6 - \dim \text{Nul } A = 6 - 5 = 1$. So $\dim \text{Col } A = \text{rank } A = 1$.

11. Since $\dim \text{Nul } A = 2$, $\text{rank } A = 5 - \dim \text{Nul } A = 5 - 2 = 3$. So $\dim \text{Row } A = \dim \text{Col } A = \text{rank } A = 3$.

12. Since $\dim \text{Nul } A = 4$, $\text{rank } A = 6 - \dim \text{Nul } A = 6 - 4 = 2$. So $\dim \text{Row } A = \dim \text{Col } A = \text{rank } A = 2$.

13. The rank of a matrix $A$ equals the number of pivot positions which the matrix has. If $A$ is either a $7 \times 5$ matrix or a $5 \times 7$ matrix, the largest number of pivot positions that $A$ could have is 5. Thus the largest possible value for $\text{rank } A$ is 5.

14. The dimension of the row space of a matrix $A$ is equal to $\text{rank } A$, which equals the number of pivot positions which the matrix has. If $A$ is either a $4 \times 3$ matrix or a $3 \times 4$ matrix, the largest number of pivot positions that $A$ could have is 3. Thus the largest possible value for $\dim \text{Row } A$ is 3.

15. Since the rank of $A$ equals the number of pivot positions which the matrix has, and $A$ could have at most 6 pivot positions, $\text{rank } A \leq 6$. Thus $\dim \text{Nul } A = 8 - \text{rank } A \geq 8 - 6 = 2$.

16. Since the rank of $A$ equals the number of pivot positions which the matrix has, and $A$ could have at most 4 pivot positions, $\text{rank } A \leq 4$. Thus $\dim \text{Nul } A = 4 - \text{rank } A \geq 4 - 4 = 0$.

17. a. True. The rows of $A$ are identified with the columns of $A^T$. See the paragraph before Example 1.
   
   b. False. See the warning after Example 2.
   
   c. True. See the Rank Theorem.
   
   d. False. See the Rank Theorem.
   
   e. True. See the Numerical Note before the Practice Problem.
Chapter 4  SUPPLEMENTARY EXERCISES

1. a. True. This set is \( \text{Span}\{v_1, \ldots, v_p\} \), and every subspace is itself a vector space.

   b. True. Any linear combination of \( v_1, \ldots, v_{p-1}, v_p \) is also a linear combination of \( v_1, \ldots, v_{p-1}, v_p \) using the zero weight on \( v_p \).

   c. False. Counterexample: Take \( v_p = 2v_1 \). Then \( \{v_1, \ldots, v_p\} \) is linearly dependent.

   d. False. Counterexample: Let \( \{e_1, e_2, e_3\} \) be the standard basis for \( \mathbb{R}^3 \). Then \( \{e_1, e_2\} \) is a linearly independent set but is not a basis for \( \mathbb{R}^3 \).

   e. True. See the Spanning Set Theorem (Section 4.3).

   f. True. By the Basis Theorem, \( S \) is a basis for \( V \) because \( S \) spans \( V \) and has exactly \( p \) elements. So \( S \) must be linearly independent.

   g. False. The plane must pass through the origin to be a subspace.

   h. False. Counterexample:

      \[
      \begin{bmatrix}
      0 & 0 & 7 & 3 \\
      0 & 0 & 0 & 0 \\
      \end{bmatrix}
      \]

   i. True. This statement appears before Theorem 13 in Section 4.6.

   j. False. Row operations on \( A \) do not change the solutions of \( Ax = 0 \).

   k. False. Counterexample: \( A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \); \( A \) has two nonzero rows but the rank of \( A \) is 1.

   l. False. If \( U \) has \( k \) nonzero rows, then rank \( A = k \) and \( \text{dimNul} \ A = n - k \) by the Rank Theorem.

   m. True. Row equivalent matrices have the same number of pivot columns.

   n. False. The nonzero rows of \( A \) span Row \( A \) but they may not be linearly independent.

   o. True. The nonzero rows of the reduced echelon form \( E \) form a basis for the row space of each matrix that is row equivalent to \( E \).

   p. True. If \( H \) is the zero subspace, let \( A \) be the \( 3 \times 3 \) zero matrix. If dim \( H = 1 \), let \( \{v\} \) be a basis for \( H \) and set \( A = [v \ v \ v] \). If dim \( H = 2 \), let \( \{u, v\} \) be a basis for \( H \) and set \( A = [u \ v \ v] \), for example. If dim \( H = 3 \), then \( H = \mathbb{R}^3 \), so \( A \) can be any \( 3 \times 3 \) invertible matrix. Or, let \( \{u, v, w\} \) be a basis for \( H \) and set \( A = [u \ v \ w] \).

   q. False. Counterexample: \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \). If rank \( A = n \) (the number of columns in \( A \)), then the transformation \( x \mapsto Ax \) is one-to-one.

   r. True. If \( x \mapsto Ax \) is onto, then \( \text{Col} \ A = \mathbb{R}^m \) and rank \( A = m \). See Theorem 12(a) in Section 1.9.

   s. True. See the second paragraph after Theorem 15 in Section 4.7.

   t. False. The \( j \)-th column of \( P_{C \to G} \) is \( b_j \).
2. The set is \( \text{Span}(S) \), where \( S = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix} \). Note that \( S \) is a linearly dependent set, but each pair of vectors in \( S \) forms a linearly independent set. Thus any two of the three vectors
\[
\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix}
\]
will be a basis for \( \text{Span}(S) \).

4. The vector \( g \) is not a scalar multiple of the vector \( f \), and \( f \) is not a scalar multiple of \( g \), so the set \( \{f, g\} \) is linearly independent. Even though the number \( g(t) \) is a scalar multiple of \( f(t) \) for each \( t \), the scalar depends on \( t \).

5. The vector \( p_1 \) is not zero, and \( p_2 \) is not a multiple of \( p_1 \). However, \( p_3 \) is \( 2p_1 + 2p_2 \), so \( p_3 \) is discarded. The vector \( p_4 \) cannot be a linear combination of \( p_1 \) and \( p_2 \) since \( p_4 \) involves \( t^2 \) but \( p_1 \) and \( p_2 \) do not involve \( t^2 \). The vector \( p_5 \) is \( (3/2)p_1 - (1/2)p_2 + p_4 \) (which may not be so easy to see at first.) Thus \( p_5 \) is a linear combination of \( p_1, p_2, \) and \( p_4 \), so \( p_5 \) is discarded. So the resulting basis is \( \{p_1, p_2, p_4\} \).

4 supplementary