Section 5.2 The Characteristic Equation

Review:

\[ A \mathbf{x} = \lambda \mathbf{x} \]

Find eigenvectors \( \mathbf{x} \) by solving \( (A - \lambda I)\mathbf{x} = \mathbf{0} \).

How do we find the eigenvalues \( \lambda \)?

\[ \begin{align*}
\mathbf{x} & \text{ must be nonzero} \\
\downarrow \\
(A - \lambda I)\mathbf{x} & = \mathbf{0} \text{ must have nontrivial solutions} \\
\downarrow \\
(A - \lambda I) & \text{ is not invertible} \\
\downarrow \\
\det(A - \lambda I) & = 0
\end{align*} \]

(called the characteristic equation)

Solve \( \det(A - \lambda I) = 0 \) for \( \lambda \) to find the eigenvalues.

Characteristic polynomial: \( \det(A - \lambda I) \)

Characteristic equation: \( \det(A - \lambda I) = 0 \)

EXAMPLE: Find the eigenvalues of \( A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \).

Solution: Since

\[ A - \lambda I = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -6 & 5 - \lambda \end{bmatrix}, \]

the equation \( \det(A - \lambda I) = 0 \) becomes

\[ -\lambda(5 - \lambda) + 6 = 0 \]

\[ \lambda^2 - 5\lambda + 6 = 0 \]

Factor:

\[ (\lambda - 2)(\lambda - 3) = 0. \]

So the eigenvalues are 2 and 3.

For a \( 3 \times 3 \) matrix or larger, recall that a determinant can be computed by cofactor expansion.
EXAMPLE: Find the eigenvalues of \( A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix} \).

Solution:

\[
A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 & 1 \\ 0 & -5 - \lambda & 0 \\ 1 & 8 & 1 - \lambda \end{bmatrix}
\]

\[
\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 2 & 1 \\ 0 & -5 - \lambda & 0 \\ 1 & 8 & 1 - \lambda \end{vmatrix} = (-5 - \lambda)[(1 - \lambda)^2 - 1] = (-5 - \lambda)[1 - 2\lambda + \lambda^2 - 1]
\]

\[
= (-5 - \lambda)[-2\lambda + \lambda^2] = -(5 + \lambda)\lambda[-2 + \lambda] = 0
\]

\[\Rightarrow \lambda = -5, 0, 2\]

THEOREM  (The Invertible Matrix Theorem - continued)

Let \( A \) be an \( n \times n \) matrix. Then \( A \) is invertible if and only if:

s. The number 0 is not an eigenvalue of \( A \).

t. \( \det A \neq 0 \)
Recall that if $B$ is obtained from $A$ by a sequence of row replacements or interchanges, but without scaling, then $\det A = (-1)^r \det B$, where $r$ is the number of row interchanges.

Suppose the echelon form $U$ is obtained from $A$ by a sequence of row replacements or interchanges, but without scaling.

$$A \sim U = \begin{bmatrix}
    u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\
    0 & u_{22} & u_{23} & \cdots & u_{2n} \\
    0 & 0 & u_{33} & \cdots & u_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & 0 & u_{nn}
\end{bmatrix}$$

The determinant of $A$, written $\det A$, is defined as follows:

$$\det A = \begin{cases}
    (-1)^r \cdot \left( \text{product of pivots in } U \right), & \text{when } A \text{ is invertible} \\
    0, & \text{when } A \text{ is not invertible}
\end{cases}$$

$(r$ is the number of row interchanges)

**EXAMPLE:** Find the eigenvalues of $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$.

**Solution:**

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & 2 & 3 \\ 0 & 6 - \lambda & 10 \\ 0 & 0 & 2 - \lambda \end{bmatrix}$$

Characteristic equation:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0. \quad \text{eigenvalues: _____, _____, _____}$$

The (algebraic) multiplicity of an eigenvalue is its multiplicity as a root of the characteristic equation.
EXAMPLE: Find the characteristic polynomial of

\[
A = \begin{bmatrix}
2 & 0 & 0 & 0 \\
5 & 3 & 0 & 0 \\
9 & 1 & 3 & 0 \\
1 & 2 & 5 & -1
\end{bmatrix}
\]

and then find all the eigenvalues and the algebraic multiplicity of each eigenvalue.

Solution:

\[
\det (A - \lambda I) = \begin{vmatrix}
2 - \lambda & 0 & 0 & 0 \\
5 & 3 - \lambda & 0 & 0 \\
9 & 1 & 3 - \lambda & 0 \\
1 & 2 & 5 & -1 - \lambda
\end{vmatrix}
\]

\[
= (2 - \lambda)(3 - \lambda)(3 - \lambda)(-1 - \lambda) = 0
\]

**eigenvalues: _____, _____, _____**

**Similarity**

Numerical methods for finding approximating eigenvalues are based upon Theorem 4 to be described shortly.

For \(n \times n\) matrices \(A\) and \(B\), we say the \(A\) is **similar** to \(B\) if there is an invertible matrix \(P\) such that

\[
P^{-1}AP = B \quad \text{or equivalently,} \quad A = PBP^{-1}.
\]

**Theorem 4:** If \(n \times n\) matrices \(A\) and \(B\) are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

**Proof:** If \(B = P^{-1}AP\), then

\[
\det(B - \lambda I) = \det[P^{-1}AP - P^{-1}\lambda IP] = \det[P^{-1}(A - \lambda I)P]
\]

\[
= \det P^{-1} \cdot \det(A - \lambda I) \cdot \det P = \det(A - \lambda I).
\]
Application to Markov Chains

**EXAMPLE**  Consider the migration matrix $M = \begin{bmatrix} .95 & .90 \\ .05 & .10 \end{bmatrix}$ and define $x_{k+1} = Mx_k$. It can be shown that

$$x_k, x_{k+1}, x_{k+2}, \ldots$$

converges to a steady state vector $x = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. Why?

The answer lies in examining the corresponding eigenvectors.

First we find the eigenvalues:

$$\det(M - \lambda I) = \det\left( \begin{bmatrix} .95 - \lambda & .90 \\ .05 & .10 - \lambda \end{bmatrix} \right) = \lambda^2 - 1.05\lambda + 0.05$$

So solve

$$\lambda^2 - 1.05\lambda + 0.05 = 0$$

By factoring

$$\lambda = 0.05, \lambda = 1$$
It can be shown that the eigenspace corresponding to $\lambda = 1$ is $\text{span}\{v_1\}$ where $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the eigenspace corresponding to $\lambda = 0.05$ is $\text{span}\{v_2\}$ where $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Note that

$$Mv_1 = v_1,$$

and so $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ is our steady state vector.

Then for a given vector $x_0$,

$$x_0 = c_1v_1 + c_2v_2$$

$$x_1 = Mx_0 = M(c_1v_1 + c_2v_2) = c_1Mv_1 + c_2Mv_2 = c_1v_1 + c_2(0.05)v_2$$

$$x_2 = Mx_1 = M(c_1v_1 + c_2(0.05)v_2) = c_1Mv_1 + c_2(0.05)Mv_2 = c_1v_1 + c_2(0.05)^2v_2$$

and in general

$$x_k = c_1v_1 + c_2(0.05)^k v_2$$

and so

$$\lim_{k \to \infty} x_k = \lim_{k \to \infty} \left( c_1v_1 + c_2(0.05)^k v_2 \right) = c_1v_1$$

and this is the steady state when $c_1 = \frac{1}{2}$. 