4.2 Null Spaces, Column Spaces, & Linear Transformations

The null space of an \( m \times n \) matrix \( A \), written as \( \text{Nul} \ A \), is the set of all solutions to the homogeneous equation \( Ax = 0 \).

\[
\text{Nul} \ A = \{x : x \text{ is in } \mathbb{R}^n \text{ and } Ax = 0\} \quad \text{(set notation)}
\]

THEOREM 2

The null space of an \( m \times n \) matrix \( A \) is a subspace of \( \mathbb{R}^n \). Equivalently, the set of all solutions to a system \( Ax = 0 \) of \( m \) homogeneous linear equations in \( n \) unknowns is a subspace of \( \mathbb{R}^n \).

Proof: \( \text{Nul} \ A \) is a subset of \( \mathbb{R}^n \) since \( A \) has \( n \) columns. Must verify properties a, b and c of the definition of a subspace.

Property (a) Show that \( 0 \) is in \( \text{Nul} \ A \). Since \( \ldots \), \( 0 \) is in \( \ldots \).

Property (b) If \( u \) and \( v \) are in \( \text{Nul} \ A \), show that \( u + v \) is in \( \text{Nul} \ A \). Since \( u \) and \( v \) are in \( \text{Nul} \ A \),

\[
\ldots \text{ and } \ldots.
\]

Therefore

\[
A(u + v) = \ldots + \ldots = \ldots + \ldots = \ldots.
\]

Property (c) If \( u \) is in \( \text{Nul} \ A \) and \( c \) is a scalar, show that \( cu \) in \( \text{Nul} \ A \):

\[
A(cu) = \ldots A(u) = c0 = 0.
\]

Since properties a, b and c hold, \( A \) is a subspace of \( \mathbb{R}^n \).
Solving $Ax = 0$ yields an **explicit description** of $	ext{Nul } A$.

**EXAMPLE:** Find an explicit description of $	ext{Nul } A$ where \( A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \).

**Solution:** Row reduce augmented matrix corresponding to $Ax = 0$:

\[
\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & | & 0 \\ 6 & 12 & 13 & 0 & 3 & | & 0 \end{bmatrix} \rightarrow \ldots \rightarrow \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & | & 0 \\ 0 & 0 & 1 & -6 & -15 & | & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix}
\]

\[
= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}
\]

Then \( \text{Nul } A = \text{span}\{u, v, w\} \).

**Observations:**

1. Spanning set of $\text{Nul } A$, found using the method in the last example, is automatically linearly independent:

\[
\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = _____ \quad c_2 = _____ \quad c_3 = _____
\]

2. If $\text{Nul } A \neq \{0\}$, the the number of vectors in the spanning set for $\text{Nul } A$ equals the number of free variables in $Ax = 0$. 

The column space of an $m \times n$ matrix $A$ (Col $A$) is the set of all linear combinations of the columns of $A$.

If $A = [a_1 \ldots a_n]$, then

$$
\text{Col } A = \text{Span}\{a_1, \ldots, a_n\}
$$

**THEOREM 3**

The column space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^m$.

Why? (Theorem 1, page 221)

Recall that if $Ax = b$, then $b$ is a linear combination of the columns of $A$. Therefore

$$
\text{Col } A = \{b : b = Ax \text{ for some } x \text{ in } \mathbb{R}^n\}
$$

**EXAMPLE:** Find a matrix $A$ such that $W = \text{Col } A$ where $W = \left\{ \begin{pmatrix} x - 2y \\ 3y \\ x + y \end{pmatrix} : x, y \in \mathbb{R}\right\}$.

**Solution:**

$$
\begin{pmatrix}
 x - 2y \\
 3y \\
 x + y
\end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}
$$

$$
= \begin{pmatrix}
 x \\
 y
\end{pmatrix}
$$

Therefore $A = \begin{pmatrix}
 x \\
 y
\end{pmatrix}$.

By Theorem 4 (Chapter 1),

The column space of an $m \times n$ matrix $A$ is all of $\mathbb{R}^m$ if and only if the equation $Ax = b$ has a solution for each $b$ in $\mathbb{R}^m$. 


The Contrast Between Nul $A$ and Col $A$

**EXAMPLE:** Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) The column space of $A$ is a subspace of $\mathbb{R}^k$ where $k =$ _____.

(b) The null space of $A$ is a subspace of $\mathbb{R}^k$ where $k =$ _____.

(c) Find a nonzero vector in Col $A$. (There are infinitely many possibilities.)

\[
\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} = \]

(d) Find a nonzero vector in Nul $A$. Solve $Ax = 0$ and pick one solution.

\[
\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 0 \\ 3 & 6 & 10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

$x_1 = -2x_2$

$x_2$ is free

$x_3 = 0$

Let $x_2 =$ ____ and then

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \end{bmatrix}
\]

**Contrast Between Nul $A$ and Col $A$ where $A$ is $m \times n$ (see page 232)**
Review

A **subspace** of a vector space $V$ is a subset $H$ of $V$ that has three properties:

a. The zero vector of $V$ is in $H$.

b. For each $u$ and $v$ in $H$, $u + v$ is in $H$. (In this case we say $H$ is closed under vector addition.)

c. For each $u$ in $H$ and each scalar $c$, $cu$ is in $H$. (In this case we say $H$ is closed under scalar multiplication.)

If the subset $H$ satisfies these three properties, then $H$ itself is a vector space.

**THEOREM 1, 2 and 3 (Sections 4.1 & 4.2)**

If $v_1, \ldots, v_p$ are in a vector space $V$, then $\text{Span}\{v_1, \ldots, v_p\}$ is a subspace of $V$.

The null space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$.

The column space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^m$.

**EXAMPLE:** Determine whether each of the following sets is a vector space or provide a counterexample.

(a) $H = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x - y = 4 \right\}$. **Solution:** Since $$ is not in $H$, $H$ is not a vector space.

(b) $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{array}{l} x - y = 0 \\ y + z = 0 \end{array} \right\}$. **Solution:** Rewrite

$$
\begin{align*}
x - y &= 0 \\
y + z &= 0
\end{align*}
$$
as

$$
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

So $V = \text{Nul } A$ where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Since $\text{Nul } A$ is a subspace of $\mathbb{R}^2$, $V$ is a vector space.
(c) \( S = \left\{ \begin{bmatrix} x + y \\ 2x - 3y \\ 3y \end{bmatrix} : x, y, z \text{ are real} \right\} \)

One Solution: Since
\[
\begin{bmatrix} x + y \\ 2x - 3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},
\]

\( S = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\} ; \) therefore \( S \) is a vector space by Theorem 1.

Another Solution: Since
\[
\begin{bmatrix} x + y \\ 2x - 3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},
\]

\( S = \text{Col} \ A \) where \( A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 3 \end{bmatrix} \); therefore \( S \) is a vector space, since a column space is a vector space.

Kernal and Range of a Linear Transformation

A linear transformation \( T \) from a vector space \( V \) into a vector space \( W \) is a rule that assigns to each vector \( x \) in \( V \) a unique vector \( T(x) \) in \( W \), such that

i. \( T(u + v) = T(u) + T(v) \) for all \( u, v \) in \( V \);
ii. \( T(cu) = cT(u) \) for all \( u \) in \( V \) and all scalars \( c \).

The kernel (or null space) of \( T \) is the set of all vectors \( u \) in \( V \) such that \( T(u) = 0 \). The range of \( T \) is the set of all vectors in \( W \) of the form \( T(u) \) where \( u \) is in \( V \).

So if \( T(x) = Ax \), \( \text{col} \ A = \text{range of} \ T. \)