Math 2410, Answer to Quiz 7 (4/6/09)

(1) Find the general solution to the equation

\[ y'' + 4y' + 20y = 2 \cos(2t) . \]

Answer: We have \( y_h = e^{\alpha t} \) being a homogeneous solution if \( \alpha^2 + 4 \alpha + 20 = 0 \). Thus \( \alpha = -2 \pm 4i \). So

\[ y_h = c_1 e^{-2t} \cos(4t) + c_2 e^{-2t} \sin(4t) . \]

Let \( y_p = A \cos(2t) + B \sin(2t) \). Putting into the equations, we have

\[ 16A + 8B = 2 , \]

\[ -8A + 16B = 0 . \]

Thus \( A = 1/10 \) and \( B = 1/20 \). Hence general solution is

\[ y = y_p + y_h = \cos(2t)/10 + \sin(2t)/20 + c_1 e^{-2t} \cos(4t) + c_2 e^{-2t} \sin(4t) . \]

(2) Find the solution to the equation

\[ y'' + 4y = \sin(2t) \]

with initial condition \( y(0) = 0 \) and \( y'(0) = 0 \).

Answer: We have \( y_h = e^{\alpha t} \) being a homogeneous solution if \( \alpha^2 + 4 = 0 \). Thus \( \alpha = \pm 2i \). So

\[ y_h = c_1 \cos(2t) + c_2 \sin(2t) . \]

Since the forcing term \( \sin(2t) \) coincides with the homogeneous solution, we need to add a \( t \) to the usual particular solution guess.

Let’s consider

\[ z'' + 4z = e^{2it} \]

Assume \( z_p = Ate^{2it} \). Upon substituting into the equation, we find that \( A = 1/(4i) = -i/4 \). Therefore \( z_p = -ite^{2it}/4 \) and

\[ y_p = \mathcal{I}m(z_p) = -t \cos(2t)/4 . \]
Thus the general solution is

\[ y = y_p + y_h = -t \cos(2t)/4 + c_1 \cos(2t) + c_2 \sin(2t) . \]

Putting in the initial conditions, we find \( c_1 = 0 \) and \( c_2 = 1/8 \). Hence

\[ y = -t \cos(2t)/4 + \sin(2t)/8 . \]