Math 2410, Answer to Quiz 4 (2/23/09)

There are 2 questions. All answers have to be accompanied by supporting calculation or reasoning.

(1) By using an integrating factor, find the general solution to

\[
\frac{dy}{dt} = -\frac{y}{1+t} + t^2
\]

Answer: First, observe that the equation is a first order linear non-homogeneous equation and the coefficient of \(\frac{dy}{dt}\) is 1. When the equation is put in the standard form \(\frac{dy}{dt} + a(t)y = g(t)\), we have \(a(t) = 1/(1+t)\). The integrating factor \(\eta\) is given by:

\[
\eta = e^{\int a(t)dt} = e^{\int \frac{dt}{1+t}} = e^{\log(1+t)} = 1 + t.
\]

Hence multiplying the equation by \((1 + t)\) on both sides, we have

\[
(1 + t)\left\{ \frac{dy}{dt} + \frac{y}{1+t} \right\} = (1 + t)t^2.
\]

Hence

\[
\frac{d}{dt}[(1 + t)y] = t^2 + t^3.
\]

Integrate the above equation to obtain:

\[
(1 + t)y = \frac{t^3}{3} + \frac{t^4}{4} + C
\]

for any constant \(C\). In other words,

\[
y = \frac{1}{1+t}\left[ \frac{t^3}{3} + \frac{t^4}{4} + C\right].
\]
(2) Find the general solution to the linear homogeneous equation $y'' - 6y' + 8y = 0$.

Answer: Let $y = e^{\alpha t}$. This will be a solution if $\alpha^2 - 6\alpha + 8 = 0$. Hence $\alpha = 4$ or $\alpha = 2$. In other words, both $y = e^{4t}$ and $y = e^{2t}$ are solutions.

Since the equation is linear and homogeneous, the general solution is

$$y = C_1 e^{4t} + C_2 e^{2t}$$

for some arbitrary constants $C_1$ and $C_2$. 