Math 211, Answer to Quiz 4 (2/20/08)

(1) (10 pts) Use the method of undetermined coefficient to solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} - y = 3e^{2t} \ , \\ \\ y(0) = 1 \ . \end{cases}$$

Answer: First we study the homogeneous equation $\dfrac{dy_h}{dt} = y_h$, which has a solution $y_h = e^t$.

Next assume a particular solution of the form $y = Ae^{2t}$. Putting it into the original non-homogeneous equation,

$$2Ae^{2t} - Ae^{2t} = 3e^{2t}$$

which gives $A = 3$. Thus the general solution is:

$$y = y_p + ky_h = 3e^{2t} + ke^t$$

for some constant k. With $y(0) = 1$, we have $1 = 3 + k$, leading to $k = -2$. As a result, the solution is

$$y = 3e^{2t} - 2e^t$$

(2) (10 pts) By using an integrating factor, find the general solution to

$$\frac{dy}{dt} = -\frac{y}{1+t} + t^2$$

Answer: Write the equation in standard form:

$$\frac{dy}{dt} + \frac{y}{1+t} = t^2$$

Observe that the equation is a first order linear non-homogeneous equation and the coefficient of dy/dt is 1. Let $a(t) = 1/(1+t)$. The integrating factor η is given by:

$$\eta = e^{\int a(t)dt} = e^{\int \frac{dt}{1+t}} = e^{\log(1+t)} = 1 + t \ .$$

Hence multiplying the equation by $(1+t)$ on both sides, we have

$$(1+t)\{\frac{dy}{dt}+\frac{y}{1+t}\}=(1+t)t^2\ .$$

Hence

$$\frac{d}{dt}[(1+t)y]=t^2+t^3\ .$$

Integrate the above equation to obtain:

$$(1+t)y=\frac{t^3}{3}+\frac{t^4}{4}+C$$

for any constant C. In other words,

$$y=\frac{1}{1+t}[\frac{t^3}{3}+\frac{t^4}{4}+C]\ .$$