Math 211 Answer to Exam 1 March, 5, 2008

(1) (20 pts) Solve the equation $y' = (1+t^2)e^{-2y}$ subject to the condition $y(0) = 1$. Express y explicitly as a function of t.

Answer: The equation is separable and it simplifies to

$$\int e^{2y} dy = \int (1+t^2) dt \ .$$

After integration, we have

$$\frac{1}{2}e^{2y} = t + \frac{t^3}{3} + C \ .$$

Put in the initial condition,

$$\frac{1}{2}e^2 = C \ .$$

Thus

$$\frac{1}{2}e^{2y} = t + \frac{t^3}{3} + \frac{e^2}{2} \ .$$

Solving for y, we obtain

$$y = \frac{1}{2}\log(2t + \frac{2}{3}t^3 + e^2) \ .$$

(2) Consider the equation

$$\frac{dy}{dt} = (y^2 - \mu)(y^2 - 4)$$

(a) (10 pts) Draw the bifurcation diagram. (Use your graphing calculator to help if necessary.)

(b) (6 pts) Put arrow directions in representative phase lines to indicate the stability of the steady state solutions.

(c) (4 pts) At what values of μ will bifurcation occur. Explain.

The answer will be posted in another page.

(3) (20 pts) Given the system

$$\begin{aligned}\frac{dx}{dt} &= y\,, \\ \frac{dy}{dt} &= x^2 + t + 1\,,\end{aligned}$$

with initial condition $x(0) = 1$ and $y(0) = 0$. Write down the general formula for the Euler's method. Use it with a step size $\Delta t = 0.1$ to find the approximate solution for x and y at $t = 0.2$.

Answer: Let $t_i = i\Delta t = 0.1\,i$ for $i = 0, 1, \ldots$, and $\mathbf{Z}_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix}$ be the numerical solution at t_i using the Euler's method. Thus $\mathbf{Z}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The Euler's method requires that

$$\mathbf{Z}_{i+1} = \mathbf{Z}_i + \Delta t \begin{pmatrix} Y_i \\ X_i^2 + t_i + 1 \end{pmatrix}$$

Hence

$$\begin{aligned}\mathbf{Z}_1 &= \mathbf{Z}_0 + \Delta t \begin{pmatrix} Y_0 \\ X_0^2 + t_0 + 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0.1 \begin{pmatrix} 0 \\ 1^2 + 0 + 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0.2 \end{pmatrix}\,.\end{aligned}$$

And

$$\begin{aligned}\mathbf{Z}_2 &= \mathbf{Z}_1 + \Delta t \begin{pmatrix} Y_1 \\ X_1^2 + t_1 + 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0.2 \end{pmatrix} + 0.1 \begin{pmatrix} 0.2 \\ 1^2 + 0.1 + 1 \end{pmatrix} \\ &= \begin{pmatrix} 1.02 \\ 0.41 \end{pmatrix}\,.\end{aligned}$$

Hence $x(0.2) \approx 1.02$ and $y(0.2) \approx 0.41$.

(4) (20 pts) Using the method of undetermined coefficients, find the general solution to the equation

$$\frac{dy}{dt} = y + \cos 2t \ .$$

Answer: The homogeneous equation is $dy_h/dt = y_h$, which has a solution $y_h = Ce^t$ for any constant C.

For the original non-homogeneous equation, let a particular solution y_p be of the form

$$y_p = A\cos 2t + B\sin 2t \ .$$

Then on putting it into the equation, we have

$$-2A\sin 2t + 2B\cos 2t = A\cos 2t + B\sin 2t + \cos 2t \ .$$

Comparing the coefficients of $\cos 2t$ and $\sin 2t$, we have

$$2B = A + 1 \qquad \text{and} \qquad -2A = B \ .$$

These yield $A = -1/5$ and $B = 2/5$. Hence $y_p = -\frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$.

The general solution will then be:

$$y = y_h + y_p = Ce^t - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t \ .$$

(5) (20 pts) Using the method of integrating factor, find the solution to

$$\frac{dy}{dt} + 2ty = t$$

with initial condition $y(0) = 1$.

Answer: With the coefficient of dy/dt being 1 already, the integrating factor is

$$e^{\int 2t\,dt} = e^{t^2} .$$

Multiplying the equation on both sides by this factor, we observe that

$$\frac{d}{dt}\{ye^{t^2}\} = te^{t^2} .$$

Hence

$$ye^{t^2} = \int te^{t^2} dt .$$

Using the substitution $u = t^2$ to perform the integration on the right, we simplify the above equation to

$$ye^{t^2} = \frac{1}{2}e^{t^2} + C$$

for some constant C. In other words,

$$y = \frac{1}{2} + Ce^{-t^2} .$$

Putting in the initial condition, we have $1 = \frac{1}{2} + C$. This gives $C = 1/2$. Therefore

$$y = \frac{1}{2} + \frac{1}{2}e^{-t^2} .$$