

*Answer to Exam 1, Math 211, Spring, 2007*

(1) Consider the equation

$$\frac{dy}{dt} = -y(y-1)(y+2)$$

(a) (3 pts) What are the equilibrium (steady state) solutions for the above equation.

*Answer:*  $y = -2$ ,  $y = 0$  and  $y = 1$  are the equilibrium solutions.

(b) (6 pts) Sketch the slope fields and, to its right, the phase lines for the equation.

*Answer is documented in another file.*

(c) (3 pts) Indicate which equilibrium points are stable or unstable. Explain.

*Answer:*  $y = 1$  and  $y = -2$  are stable (sink). Slight perturbation from their locations will decay in time.

$y = 0$  is unstable (source). Slight perturbation from its location will magnify with time.

(d) (3 pts) What happens as  $t \rightarrow \infty$  if the initial condition is  $y(0) = 1/2$ ?

*Answer:* From the phase line plot in part (b), the solution will go to  $y = 1$  as  $t \rightarrow \infty$ .

(2a) (10 pts) A 10-gallon bucket is full of pure water. Suppose we begin dumping salt into the bucket at a rate of 2 pounds per minute. Also a faucet at the bottom of the bucket is opened to let out 2 gallons of salted water per minute. At the same time, we add pure water to keep the bucket full. If the salt water is always well mixed, formulate a governing differential equation on  $S(t)$ , which is the amount of salt inside the bucket at time  $t$ . What is the initial condition on  $S$  at  $t = 0$ ? Explanation is essential.

*Answer:* The rate of salt increase to the bucket is 2 lb/min. With concentration of salt being  $S/10$  lb/gal and the salt water is being drained at 2 gal/min, the rate of salt removal is  $S/5$  lb/min. Hence the governing equation is:

$$\frac{dS}{dt} = 2 - \frac{S}{5}.$$

Since initially the tank just contains pure water, we have  $S(0) = 0$ .

(2b) (10 pts) Solve the equation  $dy/dt = 5 - 2y$  subject to  $y(0) = 1$  using the method of separation of variable.

*Answer:* We have

$$\int \frac{dy}{5 - 2y} = \int dt$$

which leads to

$$-\frac{1}{2} \log |5 - 2y| = t + C$$

where  $C$  is an arbitrary constant. This simplifies to  $5 - 2y = Ae^{-2t}$  for some constant  $A$ . Putting in the initial condition  $y(0) = 1$ , we have  $A = 3$ . Hence

$$y = \frac{5}{2} - \frac{3}{2}e^{-2t}.$$

(3) Consider the equation

$$\frac{dy}{dt} = y^3 - 9y - \mu$$

- (a) (9 pts) Draw the bifurcation diagram. (Use your graphing calculator to help.)
- (b) (4 pts) Put arrow directions in representative phase lines to indicate the stability of the steady state solutions.
- (c) (2 pts) At what value of  $\mu$  will bifurcation occur. Explain.

*Answer is documented in another file.*

(4) (15 pts) Given the system

$$\begin{cases} \frac{dx}{dt} = -2y, \\ \frac{dy}{dt} = x^2 + y. \end{cases}$$

with initial condition  $x(0) = 1$  and  $y(0) = -2$ . Write down the general formula for the Euler's method. Use it with a step size  $\Delta t = 0.5$  to approximate  $y(1)$ . What is this approximate value?

*Answer:* Let  $t_i = i\Delta t$ ,  $i = 0, 1, 2, \dots$ ,  $\vec{z} = \begin{pmatrix} x \\ y \end{pmatrix}$ ,  $\vec{Z}_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix}$  be the approximate solution to  $\vec{z}(t_i)$ . Then Euler's method amounts to

$$\vec{Z}_{i+1} = \vec{Z}_i + \Delta t \begin{pmatrix} -2Y_i \\ X_i^2 + Y_i \end{pmatrix}$$

with  $\vec{Z}_0 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ . Then

$$\vec{Z}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 0.5 \begin{pmatrix} 4 \\ 1 - 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2.5 \end{pmatrix}.$$

And

$$\vec{Z}_2 = \begin{pmatrix} 3 \\ -2.5 \end{pmatrix} + 0.5 \begin{pmatrix} 5 \\ 9 - 2.5 \end{pmatrix} = \begin{pmatrix} 5.5 \\ 0.75 \end{pmatrix}.$$

Hence  $y(1) \approx Y_2 = 0.75$ .

(5a) (10 pts) Using the method of undetermined coefficients, find the solution to the equation

$$\frac{dy}{dt} = 3y + e^{2t}$$

with initial condition  $y(0) = 0$ .

*Answer:* The homogeneous equation  $dy_h/dt = 3y_h$  has a non-trivial solution  $y_h = e^{3t}$ . For a particular solution to the non-homogeneous equation, we let  $y_p = Ae^{2t}$ . Putting it in the given equation, we have

$$2Ae^{2t} = 3Ae^{2t} + e^{2t},$$

giving  $A = -1$ . In other words,  $y_p = -e^{2t}$ . Hence the general solution is

$$y = y_p + ky_h = -e^{2t} + ke^{3t}$$

where  $k$  is an arbitrary constant. With  $y(0) = 0$ , we have  $0 = -1 + k$ . Therefore  $k = 1$ . Thus the required solution is  $y = -e^{2t} + e^{3t}$ .

(5b) (10 pts) Solve the same problem in (5a) using the method of integrating factor. Check that the solutions are the same.

*Answer:* The integrating factor is  $\eta(t) = e^{-\int 3dt} = e^{-3t}$ . Multiplying both sides of the equation by  $\eta(t)$ , we have

$$e^{-3t} \left( \frac{dy}{dt} - 3y \right) = e^{-3t} e^{2t}$$

which simplifies to

$$\frac{d}{dt}(ye^{-3t}) = e^{-t}.$$

On integration, we obtain  $ye^{-3t} = -e^{-t} + C$  for some arbitrary constant  $C$ . Hence  $y = -e^{2t} + Ce^{3t}$ . With initial condition  $y(0) = 0$ , we have  $C = 1$ . Therefore  $y = -e^{2t} + e^{3t}$ , reproducing the previous result.

(6a) (8 pts) Convert the second order equation

$$\frac{d^2x}{dt^2} + x\left(\frac{dx}{dt}\right)^2 - 1 + x^2 = 0$$

into a first-order system in term of  $x$  and  $v$ , where  $v = dx/dt$ .

*Answer:* The system is

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = -xv^2 + 1 - x^2. \end{cases}$$

(6b) (7 pts) Find all the equilibrium points of the resulting first-order system.

*Answer:* Setting the right hand of the above equations to zero, we have  $v = 0$  and  $-xv^2 + 1 - x^2 = 0$ . Hence  $x^2 = 1$  or  $x = \pm 1$ . Thus there are two equilibrium solutions  $(x, v) = (1, 0)$  and  $(x, v) = (-1, 0)$ .